Fully Revealing Rational Expectations in Financial Markets

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Abstract

Signaling and asymmetric information is incorporated into (Mirman et al., 2013). A fully revealing rational expectations equilibrium (FRREE) is defined and studied with two different assumptions about real and financial market behavior. FRREE is compared with a full information equilibrium. Comparative statics are derived. We conclude that the structure of the markets is determinant to the informational content of prices.

Keywords: Financial sector, Firm behavior, Learning, Market power, Monopoly, Nash equilibrium, Perfect competition, Publicly-traded firm, Risk aversion, Risk taking, Shareholder behavior, Signalling, Stackelberg equilibrium.

JEL Classifications: D21, D42, D82, D83, D84, L12, L15.

1 Introduction

It is well known that market prices are instrumental in disseminating information to market participants (Grossman, 1989). In (Mirman et al., 2014), the signalling role of prices is studied in a noisy environment, where there are asymmetric information among consumers, i.e., there are informed and uninformed consumers about a key charactheristic of the good being offered. The presence of noise in the market prevents complete learning by the uninformed consumers and fundamentally affects the market equilibrium.

In the present work we aim to study the effect of a similar informational asymmetry in a financial market and its consequences in the real market equilibrium. For this effect, we build on the work done in (Mirman and Santugini, 2013) and (Mirman et al., 2013), but will now assume there are two types of investors: the informed investors know the value of a key parameter that determines the firm profitability (the net output price or the net inverse demand function); the learning investors will have to infer that value from observing the market price. In our model noise is introduced only *a priori*, i.e., real market price (or demand) suffers a random shock, but the distribution of this random variable is known and incorporated in the expected utility to be maximized by all agents. The optimal behavior in equilibrium does not depend on the realization of this random variable but only on its distribution, and so complete learning may be possible. However, we will show that, depending on the structure of the markets, this learning equilibrium may differ from the full information equilibrium.

We will use the concept of Fully Revealing Rational Expectations Equilibrium (FRREE), according to which the learning investors will perfectly infer the value of the unknown parameter using an updating rule dependent on the equilibrium financial price. In equilibrium this updating rule must return the true value of the parameter.

The concept of Rational Expectations Equilibrium was first introduced in (Radner, 1979). (DeMarzo and Skiadas, 1999) stated existence conditions with risk neutral agents. More recently, some further research has been made around this equilibrium concept; see (Sun and Yannelis, 2007), (Schneider, 2009), (Ozsoylev and Werner, 2011) and (Sun and Yannelis, 2012).

Two different assumptions will be made regarding the struture of the markets: i) perfect competiton on both markets; ii) competitive real market but monopolistic financial market. We will show that different market structures imply different informational content for the financial price. This will lead to different updating rules and different partition of risk among the agents.

We consider a firm initially owned by an entrepreneur (the managing shareholder) who has the ability to issue shares of a risky asset (tied to the random profit of the firm). In our model, the entrepreneur is the deciding shareholder of the firm, he undertakes a risky project in the real sector and interacts with the remaining shareholders, the investors, in the financial sector. The project is risky because the firm faces a random price in the real market.

The allocation of risk among risk-averse shareholders is achieved by selling shares of a risky asset in the financial market. Shares of the risky asset define the ownership structure of the firm and represent claims to the profit derived in the real sector. While the entrepreneur allocates the profit of the firm among the shareholders, he retains control of the firm's decisions. Specifically, the entrepreneur decides both the level of output and the ownership structure of the firm.

For simplicity, constant absolute risk aversion (CARA) utility functions will be assumed. As risk aversion of the agents changes, so will the equilibrium change. More risk aversion by any of the agents will lead to less output and higher output prices. The increase in risk aversion of any of the investors will lead to a smaller fraction of shares floated, while the increase of risk aversion of the entrepreneur will have the opposite effect.

The impact of risk aversion on the decisions of a risk averse firm was studied as early as in Baron (1970), Baron (1971), Sandmo (1971), and Leland (1972). The relationship between real and financial sectors was nvestigated more than a decade later. Dotan and Ravid (1985), Prezas (1988), Brander and Lewis (1986) and Showalter (1995), approached this issue through the optic of optimal debt-equity allocation; Jain and Mirman (2000) approached it through the insider trading problem.

Mirman and Santugini (2013) and (Mirman et al., 2013) on risk-sharing and financial markets go much further. They use a model with a risk-averse owner of a monopolist firm (the entrepreneur) trying to float part of the stock of his firm in a financial market where there is only a risk-averse investor. To optimize his final expected utility this entrepreneur must take into account, simultaneously, his decisions on the real and on the financial market. It is this dual perspective that integrates real and financial equilibrium.

The paper is organized as follows. After this introduction Section 2 studies the Nash equilibrium with a competitive financial market, whereas Section 3 considers the Stackelberg equilibrium. We provide concluding remarks in Section 4.

2 The Basic Model with Competitive Real and Financial Markets

In this section, we present a general model combining the behavior of the firm (in the real and financial sectors) and the behavior of the shareholders. Perfect competition, i.e., prices given by the markets, is assumed for both real and financial markets. There is a random shock that affects the real market price.

Consider a firm that sells in a competitive real market[1] and has access to the financial market.[2] In the real market, the firm faces a given price (net of total cost) with a random component with known distribution and chooses the level of output $q \geq 0$. Specifically, the random price is [3] $\tilde{p}_R = \theta + \tilde{\varepsilon}$ where θ [4] is the expected inverse demand and $\tilde{\varepsilon}$ is a normally-distributed shock; the tilde sign differentiates a random variable from its realization.

[1]What we really assume is that the expected marginal profit is constant

[2]The adjective *real* refers to the sector of goods and services other than those of financial nature.

[3]The subscript R refers to the *real sector*

[4]As we consider that θ is net of all costs, thus including a normal cost of capital, it really is the extraodinary profit and can be positive or negative.

Assumption 2.1. $\tilde{\varepsilon} \sim N(0, \sigma^2)$.

The random profit of the firm is $\pi(q, \tilde{\varepsilon}) = (\theta + \tilde{\varepsilon})q$. The expected profit is thus linear in the level of output.

In the financial sector, the firm issues $S = S_0$ equity shares.[5] Each share is a claim of $\frac{1}{S_0}$ of the total profit so that each share receives a random payoff $\pi(q, \tilde{\varepsilon})/S_0$. In addition to issuing the total number of shares, the firm decides on the fraction $1 - \omega \in [0, 1]$ of the shares to be sold in the financial market at unit price P_F.[6] Hence, the variable ω defines the ownership structure of the firm, which specifies the allocation of the random profit among the shareholders.

The objective of each shareholder is to maximize the expected utility of final wealth. Each shareholder diversifies wealth between the risky asset issued by the firm and a risk-free asset. Without loss of generality, we assume that there are three types of shareholders: an entrepreneur, an informed investor and a learning investor.

The entrepreneur is the founder of the firm and the original claimant of the profit generated by his entrepreneurial prospects. The entrepreneur is also the managing shareholder of the firm, making the output decision, issuing the total number of shares, and deciding on the number of shares to be floated. Having no initial wealth[7], the entrepreneur's random final wealth is

$$\widetilde{W'_E} = \omega \cdot \pi(q, \tilde{\varepsilon}) + P_F \cdot (1 - \omega) \cdot S_0 \tag{1}$$

where $\omega \cdot \pi(q, \tilde{\varepsilon})$ is the entrepreneur's portion of the random profit of the firm and $P_F \cdot (1 - \omega) \cdot S_0$ is the wealth generated from selling $(1 - \omega) \cdot S_0$ shares at unit price P_F, and investing $P_F \cdot (1 - \omega) \cdot S_0$ in a risk-free asset with a rate of return normalized to zero.

Unlike the entrepreneur, the investors do not have entrepreneurial prospects and have no direct control over the decisions of the firm. We define two types of investors, denoted by I (the informed ones) and L (the learning ones). We

[5] Throughout this work we will assume that the number of shares issued is exogenously chosen and known to all agents.

[6] The subscript F refers to the *financial sector*.

[7] It would not change anything if he had.

embed signaling and learning in this model by letting a fraction λ ($\lambda \in [0;1]$) of investors, as well as the entrepreneur, to be informed about the distribution of the risky assets payoff, while the remaining fraction of $(1-\lambda)$ investors is not. They extract information from observing the price of the risky asset. In other words, the entrepreneur and the portion λ of the investors know the true value of θ. The remaining $(1-\lambda)$ portion of the investors are learning investors in the sense that they learn θ through the observation of the financial risky asset price P_F[8]. Specifically, a learning investor uses the updating rule $\chi(P_F)$ to learn about θ. The updating rule $\chi(P_F)$ defines the posterior beliefs of the learning buyer for any price P_F, and must be correct in equilibrium.

The investors use their initial wealth $W_I, W_L > 0$ to purchase shares of the risky asset and the risk-free asset. Hence, the investors' random final wealth is

$$\widetilde{W}_i' = W_i + \pi(q,\tilde{\varepsilon})z_i/S_0 - P_F z_i \tag{2}$$

where z_i is the number of shares purchased by the investor of type i and i=I,L. Here, $W_i - P_F z_i$ is invested in the risk-free asset and $\pi(q,\tilde{\varepsilon})z_i/S_0$ is the random payoff corresponding to z_i shares of the risky asset. Note that the return on a share of the firm is $\pi(q,\tilde{\varepsilon})/S_0 - P_F$.

Each shareholder maximizes the expected utility of final wealth defined by (1) or (2). The shareholders are assumed to be risk-averse in final wealth with constant absolute risk aversion (CARA).

Assumption 2.2. *The coefficients of absolute risk aversion are $a_E > 0$, $a_I > 0$ and $a_L > 0$ for the entrepreneur and the two types of investors, respectively.*[9] *All the coefficients of risk aversion are public knowledge.*

From (1), given that $\tilde{p}_R = \theta + \tilde{\varepsilon}$, the certainty equivalent of the entrepreneur is[10]

$$CE_E = \omega \cdot \theta + P_F \cdot (1-\omega) \cdot S_0 - a_E \sigma^2 \omega^2 q^2/2. \tag{3}$$

[8] Besides observing the financial price, we assume the investors cannot observe the output value a priori , but will deduce it from their knowledge of (or conjecture about) θ.

[9] In other words, utility functions for final wealth x are exponential: $u(x;a) = 1 - e^{-ax}, a \in \{a_E, a_I, a_L\}$.

[10] The expected utility of the entrepreneur is $\mathbb{E}u(\widetilde{W}_E; a_E) = 1 - e^{-a_E CE_E}$, where $\mathbb{E}$ is the expectation operator.

Here, $\omega \cdot \theta \cdot q + P_F \cdot (1-\omega) \cdot S_0$ is the expected payoff to the entrepreneur from the real and financial sectors weighted by the level of ownership. The term $a_E\sigma^2\omega^2q^2/2$ is the risk premium of the entrepreneur. The risk premium plays the role of a cost, due to risk aversion, imposed on the entrepreneur for bearing part of the risk.

From (2), the certainty equivalent of the informed investor is

$$CE_I = W_I + (\theta q/S_0 - P_F)z_I - a_I\sigma^2(q/S_0)^2z_I^2/2 \tag{4}$$

where $W_I+\theta q/S_0-P_F)z_I$ is the expected mean of final wealth and $a_I\sigma^2(q/S_0)^2z_I^2/2$ is the risk premium.

The certainty equivalent of the learning investor is

$$CE_L = W_L + (\chi(P_F)q/S_0 - P_F)z_L - a_L\sigma^2(q/S_0)^2z_L^2/2 \tag{5}$$

where $W_L + (\chi(P_F)q/S_0 - P_F)z_L$ is the expected mean of final wealth and $a_L\sigma^2(q/S_0)^2z_L^2/2$ is the risk premium. $\chi(P_F)$ is the updating rule the learning investor uses to determine the value of θ upon observing the financial price P_F.

2.1 Full information equilibrium

Having described the model, we now define the Full Information (FI) equilibrium with a competitive financial market. We will maintain two types of investors to facilitate the comparisions among the different scenarios. So, in this case, we assume both types of investors are perfectly informed, i.e., even the learning investor knows θ. The entrepreneur and the investors move simultaneously in a Nash equilibrium. The financial sector is perfectly competitive, i.e., the financial price is given, neither the entrepreneur nor the investors can take into account the effect of their decisions on the financial price. In equilibrium, the price of the risky asset clears the financial market by equating the quantity demanded by the investors with the quantity supplied by the firm (or the entrepreneur). The equilibrium consists of the firms' decisions made by the entrepreneur $\{q^*, \omega^*\}$, the investors' amount of shares of the risky asset demanded z_I^* and z_L^*, and the financial price P_F^*. The entrepreneur's decisions q^* have a direct effect on the investors' payoffs. However, the investors' decisions have no direct influence on the entrepreneur's payoffs. All shareholders are affected indirectly by one another through the financial price.

Definition 2.3. *The tuple $\{q^*, \omega^*, z_I^*, z_L^*, P_F^*\}$ is a Nash equilibrium with competitive real and financial market if*

1. *Given q^* and P_F^*, the informed and learning investors' quantity demanded for the risky asset are, respectively,*

$$z_I^* = \arg\max_{z \geq 0} \left\{W_I + (\theta q^*/S_0 - P_F^*)z - a_I \sigma^2 (q^*/S_0)^2 z^2/2\right\} \tag{6}$$

and

$$z_L^* = \arg\max_{z \geq 0} \left\{W_L + (\theta q^*/S_0 - P_F^*)z - a_L \sigma^2 (q^*/S_0)^2 z^2/2\right\} \tag{7}$$

2. *Given P_F^*, subject to $q \geq 0, \omega \in [0, 1]$,*

$$\{q^*, \omega^*\} = \arg\max_{q,\omega} \left\{\omega \cdot \theta q + P_F^* \cdot (1 - \omega) \cdot S_0 - a_E \sigma^2 \omega^2 q^2/2\right\} \tag{8}$$

3. *Given ω^*, z_I^*, z_L^*, $P_F^* > 0$ satisfies the market-clearing condition*

$$\lambda z_I^* + (1 - \lambda) z_L^* = (1 - \omega^*) S_0 \tag{9}$$

Proposition 2.4. *Under the conditions of Definition 2.3 there exists a Nash equilibrium with competitive real and financial markets. In equilibrium, output q^* satisfies*

$$q^* = \frac{\theta}{\omega^* a_E \sigma^2}, \tag{10}$$

the allocation of risk is defined by[11]

$$\omega^* = \frac{\frac{a_I a_L}{\lambda a_L + (1-\lambda) a_I}}{a_E + \left[\frac{a_I a_L}{\lambda a_L + (1-\lambda) a_I}\right]}, \tag{11}$$

. Moreover, the investors' quantities demanded are[12]

$$z_I^* = \frac{\theta S_0}{a_I q \sigma^2} \tag{12}$$

and

$$z_L^* = \frac{\theta S_0}{a_L q \sigma^2} \tag{13}$$

The financial price is

$$P_F^* = 0. \tag{14}$$

Proof. The first-order conditions corresponding to (8) are

$$q : \omega \cdot \theta - \omega^2 a_E \sigma^2 q = 0, \tag{15}$$
$$\omega : \theta q - P_F^* S_0 - a_E \sigma^2 \omega q^2 = 0, \tag{16}$$

evaluated at $q = q^*$ and $\omega = \omega^*$. Rearranging (15) and (16) yields

$$q : \omega \left[\theta - \omega a_E \sigma^2 q\right] = 0, \tag{17}$$
$$\omega : q \left[\theta - a_E \sigma^2 \omega q\right] = P_F^* S_0, \tag{18}$$

Solving this system we arrive at (10) and (14).

The first-order conditions corresponding to (6) and (7) , evaluated at $P_F^* = 0$ yield (12) and (13) . Next, plugging (10), (12) and (13) into the market-clearing equilibrium (9) and solving for ω^* yields (11). $\square$

[11]Notice that $\frac{a_I a_L}{\lambda a_L + (1-\lambda) a_I} = \frac{1}{\frac{\lambda}{a_I} + \frac{(1-\lambda)}{a_L}}$ is an inversed average risk aversion coefficient, i.e., the lower the risk coefficient of a type of consumer the more weight it will have in the average risk aversion. If $a_I < a_L$, then $a_I < \frac{1}{\frac{\lambda}{a_I} + \frac{(1-\lambda)}{a_L}} < a_L$ and vice-versa; if they are equal, the inversed average will also be equal; if there is only one type of consumers (λ equals 0 or 1), the inversed average will be equal to the risk coefficient of the only type of consumer.

[12]The investors cannot observe q, but know it is given by (10)

The results of Proposition 2.4 are in line with previous literature, namely Mirman et al. (2013), except for the fact that financial price is zero in equilibrium. The entrepreneur seems to give away part of his shares to compensate for the extra risk of increasing production. This strange behavior is rational if we remember that θ is expected real price net of all costs; in this total cost should be included the normal return on capital (the interest rate of the riskless asset). Hence, θ really represents an extraordinary profit and P_F is a financial premium over the normal market price. When there is a negative shock in the real market, the extraordinary profit may become negative, meaning that the investors will receive a dividend below normal and P_F becomes negative. $P_F = 0$ means that the stock is sold without a premium, at the normal financial price.[13]

Optimal output is reached when marginal profit (θ) equals marginal risk cost ($\omega^* a_E \sigma^2 q$). At this point, the marginal value of the firm for the entrepreneur is zero. The entrepreneur can increase this marginal value if he decreases ω, even at zero financial price. Decreasing ω he can increase output and, because the real price is fixed, there is a strong incentive to increase production, if the entrepreneur can shed away part of the extra risk. By increasing q and lowering ω he can compensate the two effects.

It is easy to see that output decreases when risk (σ^2) or any of the risk aversion coefficients increases.

In equilibrium $\mathrm{E}\pi(q,\tilde{\varepsilon}) = \theta^2/a_E\sigma^2\omega$. $CE_E = \theta^2/2a_E\sigma^2$, while $CE_I = W_I + (\theta q^*/S_0)\,z - a_I\sigma^2(q^*/S_0)^2 z^2/2 = W_I + \theta^2/2a_I\sigma^2$ and $CE_L = W_L + \theta^2/2a_L\sigma^2$

In the end he will have the same CE_E as if there was no risk sharing, but more output and more profit. When $\omega = 1$, $CE_E = \theta^2/2a_E\sigma^2$, $q^* = \frac{\theta}{a_E\sigma^2} \leq \frac{\theta}{\omega^* a_E\sigma^2}$, and $\mathrm{E}\pi(q,\tilde{\varepsilon}) = \theta^2/a_E\sigma^2 \leq \theta^2/a_E\sigma^2.\omega$. All the extra profit goes to increase the CE of the investors, to compensate them for their share of the risk.

[13] An equivalent way of interpreting this situation is to assume that the firm is financed in a kind of *'project finance'*, i.e, all its capital is borrowed. In this case, the normal return on capital is a cost paid to the lenders, while the owners of the shares will only receive a dividend if there is an extraordinary profit. Without any extraordinary profit the financial value of each share is zero.

$$\theta^2\left[\frac{1}{a_E\sigma^2\omega}-\frac{1}{a_E.\sigma^2}\right]=\frac{\theta^2}{a_E.\sigma^2}\left[\frac{1-\omega}{\omega}-\right]=\lambda\left[\frac{\theta^2}{a_I\sigma^2}\right]+(1-\lambda)\left[\frac{\theta^2}{a_L\sigma^2}\right] \tag{19}$$

There is no equilibrium with $P_F^* > 0$. If the financial price were strictly positive the entrepreneur could increase his CE indefinitely by increasing production and simultaneously shifting the risk to the investor.

2.2 Rational Expectations equilibrium

In this subsection we will analyze the same model as in the previous subsection, but with an asymmetry in information between the two types of investors: the informed type will know the true value of θ, while the learning type will have to deduce it by observing the financial price. We start by defining the fully revealing rational expectations equilibrium (FRREE). We assume that the learning investor will update his knowledge about the unknown real market parameter (in this case, the expected price θ) upon observing the financial price and that this update is perfectly accurate, i.e., $\chi(P_F) = \theta$.

The equilibrium consists of the firms' decisions made by the entrepreneur $\{q^*, \omega^*\}$, the amount of shares of the risky asset demanded by both investor's, z_I^* and z_L^*, the learning investor updating rule $\chi(P_F)$ and the financial price P_F^*.

Definition 2.5. *The tuple $\{q^*, \omega^*, z_I^*, z_L^*, P_F^*, \chi^*(P_F)\}$ is a fully revealing rational expectations equilibrium with competitive real and financial market if*

1. *Given q^* and P_F^*, the informed and learning investors' quantity demanded for the risky asset are, respectively,*

$$z_I^* = \arg\max_{z\geq 0} \left\{W_I + (\theta q^*/S_0 - P_F^*)z - a_I\sigma^2(q^*/S_0)^2 z^2/2\right\}. \tag{20}$$

and

$$z_L^* = \arg\max_{z\geq 0} \left\{W_L + (\chi(p_F)q^*/S_0 - p_F^*)z - a_L\sigma^2(q^*/S_0)^2 z^2/2\right\}. \quad (21)$$

2. *Given* P_F^*, *subject to* $q \geq 0, \omega \in [0,1]$,

$$\{q^*, \omega^*\} = \arg\max_{q,\omega} \left\{\omega \cdot \theta q + P_F^* \cdot (1-\omega) \cdot S_0 - a_E\sigma^2\omega^2 q^2/2\right\}. \quad (22)$$

3. *Given* $\omega^*, z_I^*, z_L^*, \chi(P_F^*)$, $P_F^* > 0$ *satisfies the market-clearing condition*

$$\lambda z_I^* + (1-\lambda)z_L^* = (1-\omega^*)S_0. \quad (23)$$

4. $\chi(P_F^*) = \theta$

Before characterizing the FRRE equilibrium we comment on the updating rule used by the learning investors. Indeed, learning investors form expectations about the relationship between the financial price P_F^* and the expected return of the risky asset θ. Such expectations constitute an updating rule specifying the posterior beliefs about θ for any price. In this paper we assume that learning investors use their knowledge of the structure of the economy in the full information case to form expectations. In the full information case, the number of shares floated, $(1-\omega^*)S_0$, is proportional to S_0 and thus each learning investor forms expectations about the monopolists behavior which are consistent with the full information case.

Assumption 2.6. *The updating rule* $\chi(P_F)$ *is consistent with the market clearing condition and the firms strategy in the full information case, i.e., for all* P_F, $\chi(P_F)$ *satisfies*

$$\lambda\frac{\chi(P_F)\,(q/S_0) - p_F}{a_I\sigma^2\,(q/S_0)^2} + (1-\lambda)\frac{\chi(P_F)\,(q/S_0) - P_F}{a_L\sigma^2\,(q/S_0)^2} = (1-\omega^*)S_0, \quad (24)$$

Proposition 2.7. *Under the conditions of Definition 2.5 and Assumption 2.6 there exists a Nash equilibrium with competitive real and financial markets.*

In equilibrium, output q^* satisfies

$$q^* = \frac{\theta}{\omega^* a_E \sigma^2}$$

(25)

the allocation of risk is defined by

$$\omega^* = \frac{a_I}{a_I + \lambda a_E} \tag{26}$$

Moreover, the financial price is

$$P_F^* = 0 \tag{27}$$

the informed investor's quantity demanded is

$$z_I^* = \frac{\theta S_0}{a_I q \sigma^2} \tag{28}$$

and the learning investor's quantity demanded is

$$z_L^* = 0 \tag{29}$$

Finally,

$$\chi(P_F^*) = 0 \tag{30}$$

Proof. The first-order conditions corresponding to problem (22) are

$$q : \omega \cdot \theta - \omega^2 a_E \sigma^2 q = 0, \tag{31}$$

$$\omega : \theta q - P_F^* S_0 - a_E \sigma^2 \omega q^2 = 0, \tag{32}$$

evaluated at $q = q^*$ and $\omega = \omega^*$. Rearranging (31) and (32) yields

$$q : \omega \left[\theta - \omega a_E \sigma^2 q\right] = 0, \tag{33}$$

$$\omega : q \left[\theta - a_E \sigma^2 \omega q\right] = P_F^* S_0, \tag{34}$$

Solving this system we arrive at (25) and (27).

The first-order condition corresponding to (20) and (21), evaluated at $P_F^* = 0$ yields (28) and (29). Next, plugging (28), (29) and (25) into (24) and solving for $\chi(P_F)$ yields [14]

$$\chi(P_F)^2 = \frac{\left[\lambda a_L + (1-\lambda) a_I\right] S_0}{d\left[\lambda a_L + (1-\lambda) a_I - d c a_I a_L \sigma^2\right]} P_F^* \tag{35}$$

Substituting (28), (29), (35) into (23) yields (26). Finally, the rule (35), evaluated at (25) and (27) yields (30). □

From 2.7 we can see that output with asymmetric information is smaller than with perfect information. Note that ω in (26), $\omega^* = \frac{a_I}{a_I + \lambda a_E}$, is bigger than in (11), $\omega^* = \frac{\frac{a_I a_L}{\lambda a_L + (1-\lambda) a_I}}{a_E + \left[\frac{a_I a_L}{\lambda a_L + (1-\lambda) a_I}\right]} = \frac{a_I}{a_I + \lambda a_E + (1-\lambda)\frac{a_I a_E}{a_L}}$. Larger ω implies a smaller q. Again, it is easy to see that output decreases when risk (σ^2) or any of the risk aversion coefficients increases.

It is obvious that this Nash equilibrium is not a FRRE equilibrium, because the learning investor cannot learn the true value of the parameter θ. As in the full information case, the entrepreneur sells part of his shares at $P_F = 0$ to compensate for the extra risk of increasing production. But, as the price is not informative (it does not depend on θ) and there is no other way for the learning investor to infer the value of θ, his financial demand is zero. Only the informed investor is willing to accept some shares and bear part of the risk.

The fact that the observed financial price conveys no information is quite important,because it is a common result in competitive markets. With perfect competition, equilibrium tends to be consistent with zero economic profit

[14] Again, the learning investors cannot observe q, but assume it is given by $q = \frac{\chi(P_F)}{\omega^* a_E \sigma^2} = d.\chi(P_F)$.

for the firms, driving financial price of shares also to zero, independently of any other known or unknown parameter. So, financial price cannot be used as a signal to infer any unknown information.

This is not the case in monopolistic markets, as we will see in the next section.

3 The Model with Monopolistic Financial Markets

The model in this section is basically the same as in section 2. The only (but quite important) difference pertains to the behavior of the entrepreneur in the financial market: instead of being a price-taker, we now assume he will act as a leader in the Stackelberg sense. The entrepreneur knows the financial demand functions of the investors and incorporates these functions in his Certainty Equivalent before maximizing it. In effect, the entrepreneur manipulates the financial price, P_F, to optimize his final wealth. This manipulation will be a crucial source of information in the market.

This kind of asymmetry seems to fit well to some financial operations, for instance when a firm launches an Initial Public Offer (IPO).

3.1 Full information equilibrium

We can now define the Full Information (FI) equilibrium with a monopolistic financial market. All agents (entrepreneur and both types of investors) know θ. Asymmetry in the financial market arises from the fact that the entrepreneur acts as a Stackelberg leader while both investors act as price-taking followers.

In this case the equilibrium consists of the firms' decisions made by the entrepreneur $\{q^*, \omega^*\}$, the investors' amount of shares of the risky asset demanded z_I^* and z_L^*, and the financial price P_F^*.

Definition 3.1. *The tuple $\{q^*, \omega^*, z_I^*, z_L^*, P_F^*\}$ is a Stackelberg equilibrium with competitive real market and a monopolistic financial market if*

1. *Given q^* and P_F^*, the informed and learning investors' quantity demanded for the risky asset are, respectively,*

$$z_I^*(q^*, P_F^*) = \arg\max_{z \geq 0} \left\{ W_I + (\theta q^*/S_0 - P_F^*)z - a_I \sigma^2 (q^*/S_0)^2 z^2/2 \right\} \tag{36}$$

and

$$z_L^*(q^*, P_F^*) = \arg\max_{z \geq 0} \left\{ W_L + (\theta q^*/S_0 - P_F^*)z - a_L \sigma^2 (q^*/S_0)^2 z^2/2 \right\} \tag{37}$$

2. *Given $z_I^*(q^*, P_F^*)$ and $z_L^*(q^*, P_F^*)$, subject to $q \geq 0, \omega \in [0, 1]$,*

$$\{q^*, \omega^*\} = \arg\max_{q,\omega} \left\{ \omega \cdot \theta q + D^*(q, \omega) \cdot (1 - \omega) \cdot S_0 - a_E \sigma^2 \omega^2 q^2/2 \right\} \tag{38}$$

where $P_F = D^*(q, \omega)$ is the inverse financial demand defined by (39)

3. *Given ω^*, z_I^*, z_L^*, $P_F^* > 0$ satisfies the market-clearing condition*

$$\lambda z_I^*(q^*, P_F^*) + (1 - \lambda) z_L^*(q^*, P_F^*) = (1 - \omega^*) S_0 \tag{39}$$

Proposition 3.2. *Under the conditions of Definition 3.1 there exists a equilibrium with competitive real market and monopolistic financial market. In equilibrium, output q^* satisfies*

$$q^* = \frac{\theta}{\omega^* a_E \sigma^2}$$

(40)

the allocation of risk is defined by

$$\omega^* = \frac{\frac{2a_I a_L}{\lambda a_L + (1-\lambda) a_I}}{a_E + \left[\frac{2a_I a_L}{\lambda a_L + (1-\lambda) a_I}\right]} \tag{41}$$

The financial price is

$$p_F^* = \frac{\theta}{2}.\frac{q^*}{S_0} \tag{42}$$

Moreover, the investors' quantities demanded are

$$z_I^* = \frac{\theta S_0}{2a_I q\sigma^2} \tag{43}$$

and

$$z_L^* = \frac{\theta S_0}{2a_L q\sigma^2} \tag{44}$$

Proof. The first-order conditions corresponding to (36) and (37) lead to

$$z_I^* = \frac{\theta \frac{q}{S_0} - P_F^*}{a_I\sigma^2(\frac{q}{S_0})^2} \tag{45}$$

and

$$z_L^* = \frac{\theta \frac{q}{S_0} - P_F^*}{a_L\sigma^2(\frac{q}{S_0})^2} \tag{46}$$

Substituting (45) and (46) in (39) we arrive at

$$P_F^*(q,\omega) = \left[\theta - \frac{(1-\omega)a_I a_L \sigma^2 q}{\lambda a_L + (1-\lambda)a_I}\right].\frac{q}{S_0} \tag{47}$$

Substituting (47) into (38), finding the FOCs and simplifying, we arrive at (41) and (40).

Substituting (41) and (40) in (47) we arrive at (42).

Substituting (42) into (45) and (46) we arrive at (43) and (44).

Finally, (39) can be verified. □

It is easy to see that output decreases when risk (σ^2) or any of the risk aversion coefficients increases. Also, output is smaller than with full information in a competitive financial market (subsection 2.1).

With a leading entrepreneur the financial price is an increasing function of quality, it does not have to be zero. There may be an equilibrium with positive financial price, because the entrepreneur knows that when selling more shares the financial price decreases; so, it is no longer possible to increase output indefinitely, while spreading the risk by selling more shares.

By manipulating the supply of shares in the market, the entrepreneur can retain part of the extra profit obtained with the increase in production allowed by sharing part of the risk.

In equilibrium $\mathrm{E}\pi(q,\tilde{\varepsilon}) = \frac{\theta^2}{a_E\sigma^2\omega} > \frac{\theta^2}{a_E\sigma^2}$ and the increase in profit comparing to the no risk-sharing situation is $\frac{(1-\omega)\theta^2}{a_E\sigma^2.\omega}$.

The CE_E is $\frac{\theta^2}{2a_E\sigma^2} + \frac{(1-\omega)\theta^2}{2a_E\sigma^2.\omega}$. So, the increase in the CE_E, and in the profit that goes to the entrepreneur, comparing with the no risk-sharing situation, is $\frac{(1-\omega)\theta^2}{2a_E\sigma^2\omega}$, the revenue from floating part of the shares and exactly half the increase in profit. The other half will be split bettween the two types of investors.

$$CE_I = W_I + \tfrac{\theta^2}{2a_I\sigma^2} - \tfrac{\theta^2}{8a_I\sigma^2} = W_I + \tfrac{3\theta^2}{8a_I\sigma^2} \text{ and } CE_L = W_L + \tfrac{3\theta^2}{8a_L\sigma^2}$$

So the increase in profit that goes to the investors is

$$\lambda\theta^2 2a_I\sigma^2 + (1-\lambda)\theta^2 2a_L\sigma^2 = \tfrac{(1-\omega)\theta^2}{2a_E\sigma^2.\omega}$$

.

3.2 FRRE equilibrium

In this subsection we define the Fully Revealing Rational Expectations equilibrium (FRREE) with a monopolistic financial market. The assumptions are the same as in subsection 2.2, but for the fact that the entrepreneur acts a leader in the financial market. This will make all the difference in informational content of the financial price, ensuring the existence of a FRRE equilibrium.

The entrepreneur acts as a Stackelberg leader while both investors act as price-taking followers.The entrepreneur and the informed investors know θ, while the learning type will have to deduce it by observing the financial price. We start by defining the FRREE. We assume that the learning investor will update his knowledge about the unknown real market parameter (in this case, the expected price θ) upon observing the financial price and that this update is perfectly accurate, i.e., $\chi(P_F) = \theta$.

In this case, the equilibrium consists of the firms' decisions made by the entrepreneur $\{q^*, \omega^*\}$, the amount of shares of the risky asset demanded by both investors, z_I^* and z_L^*, the learning investor updating rule $\chi(P_F)$ and the financial price P_F^*.

Definition 3.3. *The tuple* $\{q^*, \omega^*, z_I^*, z_L^*, P_F^*, \chi^*(P_F)\}$ *is a fully revealing rational expectations equilibrium with competitive real market and a monopolistic financial market if*

1. *Given* q^* *and* P_F^**, the informed and learning investors' quantity demanded for the risky asset are, respectively,*

$$z_I^*(q^*, P_F^*) = \arg\max_{z\geq 0} \left\{W_I + (\theta q^*/S_0 - P_F^*)z - a_I\sigma^2(q^*/S_0)^2 z^2/2\right\} \tag{48}$$

   and

$$z_L^*(q^*, P_F^*) = \arg\max_{z\geq 0} \left\{W_L + (\chi(P_F)q^*/S_0 - P_F^*)z - a_L\sigma^2(q^*/S_0)^2 z^2/2\right\} \tag{49}$$

2. *Given* $z_I^*(q^*, P_F^*)$ *and* $z_L^*(q^*, P_F^*)$*, subject to* $q \geq 0, \omega \in [0,1]$*,*

$$\{q^*, \omega^*\} = \arg\max_{q,\omega} \left\{\omega \cdot \theta q + D^*(q,\omega) \cdot (1-\omega) \cdot S_0 - a_E\sigma^2\omega^2 q^2/2\right\} \tag{50}$$

   where $P_F = D^*(q,\omega)$ *is the inverse financial demand defined by* (51)

3. *Given* ω^*, z_I^*, z_L^**,* $P_F^* > 0$ *satisfies the market-clearing condition*

$$\lambda z_I^*(q^*, P_F^*) + (1-\lambda) z_L^*(q^*, P_F^*) = (1-\omega^*)S_0 \tag{51}$$

4. $\chi(p_F^*) = \theta$

Proposition 3.4. *Under the conditions of Definition 3.3 there exists a fully revealing rational expectations equilibrium with competitive real market and monopolistic financial market. In equilibrium, output* q^* *satisfies*

$$q^* = \frac{\theta}{\omega^* a_E \sigma^2} \tag{52}$$

the allocation of risk is defined by

$$\omega^* = \frac{\frac{2a_I a_L}{\lambda a_L + (1-\lambda)a_I}}{a_E + \left[\frac{2a_I a_L}{\lambda a_L + (1-\lambda)a_I}\right]} \tag{53}$$

The financial price is

$$p_F^* = \frac{\theta}{2}.\frac{q^*}{S_0} \tag{54}$$

Moreover, the investors' quantities demanded are

$$z_I^* = \frac{\theta S_0}{2a_I q\sigma^2} \tag{55}$$

and

$$z_L^* = \frac{\chi(p_F^*)S_0}{2a_L q\sigma^2} \tag{56}$$

and the learning investor's updating rule is

$$\chi(P_F^*)^2 = \frac{[\lambda a_L + (1-\lambda)a_I]\, S_0}{d.\,[\lambda a_L + (1-\lambda)a_I] - d.(1-\omega)a_I a_L \sigma^2} P_F^* = \theta^2 \tag{57}$$

where $q = d.\chi(P_F^*)$.

Proof. The first-order conditions corresponding to (48) and (49) lead to

$$z_I^* = \frac{\theta\frac{q}{S_0} - P_F^*}{a_I\sigma^2(\frac{q}{S_0})^2} \tag{58}$$

and

$$z_L^* = \frac{\chi(P_F^*)\frac{q}{S_0} - P_F^*}{a_L\sigma^2(\frac{q}{S_0})^2} \tag{59}$$

Next, plugging (58) and (59) into (24) and solving for $\chi(p_F)$ yields

$$\chi(P_F)^2 = \frac{[\lambda a_L + (1-\lambda)a_I]\, S_0}{d\,[\lambda a_L + (1-\lambda)a_I - dca_I a_L\sigma^2]} P_F^* \tag{60}$$

. Substituting (58), (59) and (60) in (51) we arrive at

$$P_F^*(q,\omega) = \left[\theta - \frac{(1-\omega)a_I a_L \sigma^2 q}{\lambda a_L + (1-\lambda)a_I}\right] . \frac{q}{S_0} \tag{61}$$

Substituting (61) into (50), finding the FOCs and simplifying, we arrive at (53) and (52). Substituting (53) and (52) in (61) we arrive at (54). Substituting (54), (53) and (52) into (60) we get (57), $\chi(P_F^*) = \theta$. Substituting (54) and (57) into (58) and (59) we arrive at (55) and (56). Finally, (51) can be verified. □

With the entrepreneur acting as a leader in the financial market, the FR-REE is exactly the same as the FI equilibrium. The learning investor reaches perfect knowledge about θ, meaning that his demand is exactly the same as in the full information case. But perfect learning is only possible because the financial price contains enough information about the quality parameter, what did not happen in the competitive financial market situation.

4 Final Remarks

We computed the Full Information and the Fully Revealing Rational Expectations equilibria in a model where a firm producing for a competitive real market tried to float part of its shares in the financial market. In the financial market we considered two types of investors: a perfectly informed investor and another investor that had to infer the value of a crucial firm's profit parameter upon observing the financial price.

Two different financial market situations were considered: in the first the firm was a price taker, just like the investors; in the second the firm was a price leader, while the investors were price followers.

We reached four main conclusions:

1. With competitive real and financial markets, the financial price does not convey enough information to allow the existence of a FRRE equilibrium. The learning investor cannot infer the true value of the unknown quality parameter, reducing his financial demand to zero.

2. With competitive real market but a monopolistic financial market, i.e., with the entrepreneur of the firm being able to manipulate the financial price to optimize his expected utility, the financial price conveys enough information to ensure the existence of a FRRE equilibrium. In this case, the FRREE is exactly the same as the FI equilibrium

3. In all the scenarios, output decreases with the increase in uncertainty in the real market or in the risk aversion of any of the agents. At the same time, the fraction of shares floated does not depend on uncertainty in the real market; it increases when risk aversion of the entrepreneur also increases, and decreases when the weighted average of the investor's risk aversion increases. Output is larger in FI equilibrium with competitive market than with monopolistic financial market, while (and because) the fraction of shares floated is also larger.

4. The financial price, i.e. the premium over the normal financial return, with competitive financial market is zero. With monopolistic financial market the financial price is positive, increasing with the profitability parameter and with the level of output.

We are pursuing further investigation in this field, considering different assumptions on the structure of the markets, namely the effect of some market power by the firm in the real market. Changes in the real market also have consequences upon the informational content of the financial price.

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