Discrete dynamics for the core-periphery model

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Abstract

It is known that the discretization of continuous-time models can introduce chaotic behaviour, even when this is not consistent with observations or even the model's assumptions. We propose generic dynamics describing discrete-time core-periphery models that comply with the established assumptions in the literature and are consistent with observed behaviour. The desired properties of the dynamics are proved analytically in the general case. We also give particular forms for the dynamics for those interested in applying our model.

Keywords: Core-periphery, discrete-time, dynamics

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1 Introduction

Since Krugman (1991) set up a seminal core-periphery model, this has been studied and adapted to provide deeper insights into the patterns of industry location and the dynamics associated to it. Surprisingly enough, given the present era of computer simulations, most of the developments and extensions of the model have kept the temporal framework to continuous-time.

To the best of our knowledge, Currie and Kubin (2006)[2] provide the first discrete-time version of the core-periphery model. The dramatic appearance of chaos in this model explains, perhaps, the fact that no developments by other authors succeeded this first attempt at discretization. There are, however, further developments of this discrete-time framework, all of which bring cautionary remarks concerning unpredictability and chaos: see Commendatore *et al.* (2008) for a description of catastrophic effects of "minute changes in a tax or a subsidy", as well as the extension to three regions by Commendatore and Kubin (2013). In Currie and Kubin (2006) the effects of chaotic dynamics are so extreme that workers are seen to settle in the region providing the lowest utility, thus contradicting the assumptions of the model.

Although the existence of chaos in the discrete-time models mentioned above is beyond a doubt, we argue that this is not a universal feature of discrete-time core-periphery models. To prove our point, we propose a new discrete-time model whose dynamics not only predict an analogous population distribution to the continuous-time models, but also do not contradict the assumptions of the model. In particular, workers always settle in the region offering the highest real wages. The simplicity of the model allows for a comprehensive analytic study of the stability of all steady-state configurations, including asymmetric ones. Besides, using simulations we are able to enquire into the details of the dynamics, namely the speed of convergence for relevant values of the transportation cost. Other than providing and analysing a general version of such a discrete-time model, we propose a list of particular modelling alternatives which adapt to more precise properties. We hope to contribute to the further development of New Economic Geography by providing a model which is easy to implement for numerical analysis, easy to extend to more regions and avoids the dangers of chaos.

In the next section we give a very brief presentation of the model, under the assumption that most readers will be familiar with the standard construction. We provide references for those readers requiring more detail. Section 3 looks at the sources and consequences of chaos in the model of Currie and

[2]We are aware of a previous article by Yokoo (2001) where the dynamics of a discrete-time model are studied but the model does not take into account the usual microeconomic foundations.

Kubin (2006). In Section 4, a new model is proposed and analysed. This is done in three subsections addressing, in this order, the generic properties of the model, its steady-states and their stability, and the new dynamcis exhibited by this new model. The longer and/or more technical proofs are given in an appendix, as are the particular modelling alternatives. Section 5 concludes.

2 The core-periphery model: a brief description

The modern core-periphery model dates back to Krugman (1991). We consider the slightly different assumptions of Fujita *et al.* (1999) as used in the discrete version proposed by Currie and Kubin (2006). Below is a very short description of the model which should suffice for readers familiar with the literature. Readers looking for more detail can find it in the previously mentioned references.

There are two regions (1 and 2) and two sectors. As usual, we refer to the two sectors as industry or manufactures and agriculture. Production factors in each sector are given by the workers which are assumed mobile in the industry, and immobile and evenly distributed in the two regions in agriculture. Denote by F the total population in the agricultural sector and by L the total population in the industrial sector. We also assume that workers do not change sector. The product of the agricultural sector is assumed free of transportation costs and its price is used as numeraire. The price of the product of the industrial sector is affected by transportation costs between regions which are of *iceberg* type, represented by $T > 1$, and describing the portion of product that arrives at the destination when one unit is shipped. The transportation cost is paid by the consumer. Consumers share Cobb-Douglas preferences for the consumption of the agricultural product and for an aggregate of industrial product varieties. The aggregate is determined by a CES function. Consumption is influenced by the proportion of income spent in industrial varieties, $\mu \in [0, 1]$, and the love of variety, $\sigma > 1$.

Thus, the model depends on three parameters, namely, T, μ and σ. These influence the solution of the model and, hence, the contribution made by the model to a better understanding of the location of industry. Krugman's (1991) insights were obtained by fixing μ and σ while varying T. Subsequent research has focussed on the influence of transportation costs, represented in various forms: $\tau > 1$, instead of T, or $\phi \in (0, 1)$, representing the degree of market integration given by $\phi = \tau^{1-\sigma}$. For completion we present next the equations describing the short-run equilibrium, which we label (1), (2) and (3). When required, we use the double index r, t, where $r = 1, 2$ refers to each region and t denotes an instant in time. We denote by $\lambda \in [0, 1]$ the fraction of industrial population in region 1. Since the population is assumed constant over time, the fraction of industrial population in region 2 is given by $1 - \lambda$. The nominal wages in each region are represented by the letter w and the real wages by ω.

$$\begin{aligned} w_{1,t} =& \frac{\sigma - 1}{\beta\sigma} \left[\frac{\mu\beta}{\alpha(\sigma - 1)} \right]^{\frac{1}{\sigma}} \left[Y_{1,t} G_{1,t}^{\sigma-1} + Y_{2,t} G_{2,t}^{\sigma-1} T^{1-\sigma} \right]^{\frac{1}{\sigma}} \\ w_{2,t} =& \frac{\sigma - 1}{\beta\sigma} \left[\frac{\mu\beta}{\alpha(\sigma - 1)} \right]^{\frac{1}{\sigma}} \left[Y_{1,t} G_{1,t}^{\sigma-1} T^{1-\sigma} + Y_{2,t} G_{2,t}^{\sigma-1} \right]^{\frac{1}{\sigma}} \end{aligned} \tag{1}$$

where, for $r = 1, 2$, the price index, $G_{r,t}$, and the nominal income, $Y_{r,t}$, are given by

$$\begin{aligned} G_{1,t} =& \frac{\beta\sigma}{\sigma - 1} \left(\frac{L}{\alpha\sigma} \right)^{\frac{1}{1-\sigma}} \left[\lambda_t w_{r,t}^{1-\sigma} + (1 - \lambda_t) w_{2,t}^{1-\sigma} T^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \\ G_{2,t} =& \frac{\beta\sigma}{\sigma - 1} \left(\frac{L}{\alpha\sigma} \right)^{\frac{1}{1-\sigma}} \left[\lambda_t w_{1,t}^{1-\sigma} T^{1-\sigma} + (1 - \lambda_t) w_{2,t}^{1-\sigma} \right]^{\frac{1}{1-\sigma}}, \\ Y_{1,t} =& \frac{F}{2} + w_{1,t} \lambda_t L, \\ Y_{2,t} =& \frac{F}{2} + w_{2,t} (1 - \lambda_t) L. \end{aligned} \tag{2}$$

The real wages are then calculated according to

$$\omega_{r,t} = \frac{w_{r,t}}{G_{r,t}^{\mu}}. \tag{3}$$

The constants α and β, related to the firms' cost, are not relevant in the sequel (see below).

In what concerns the long-run equilibrium, all models used in the study of industry location assume that the decision to move from one region to another is made by workers based solely on a comparison between their own welfare measured by the real wages in each region. Thus, industrial workers migrate if and only if their current real wages are smaller than the available real wages in the other region. This assumption is preserved by models where migration dynamics are introduced. See Forslid and Ottaviano (2003) for continuous-time and Currie and Kubin (2006) for discrete-time dynamics. A distribution of population is an equilibrium if no industrial worker has an incentive to migrate. Hence, it is expected that at any long-run equilibrium the real wages are identical in both regions.

Common equilibria are symmetric dispersion, when the industrial population is evenly distributed in the two regions, that is, $\lambda = 1/2$, and concentration (agglomeration or core-periphery), when the industrial population is concentrated in a single region (the core) leaving the other region with just agricultural population (the periphery). In this case, $\lambda = 1$ when concentration occurs in region 1 or $\lambda = 0$ when concentration occurs in region 2.

The comparison between real wages in the two regions can be made either using the difference or the quotient of the real wages. As Currie and Kubin (2006), we use the latter and define

$$R(\lambda_t) = \frac{\omega_{1,t}}{\omega_{2,t}}. \tag{4}$$

Note that the parameters α and β, related to the cost incurred by firms, present in the expression of the real wages disappear in that of the quotient. When $R(\lambda_t) > 1$, workers in region 2 want to migrate to region 1 since, in region 1, real wages are higher. Migration dynamics must then be modelled in a way that reflects this assumption, such that the fraction of population in each region at time $t+1$ depends on the fraction of population in each region at the previous instant of time, t, and also on the ratio (4).

The model used by Currie and Kubin (2006) is constructed using Puga (1998) and is as follows

$$\lambda_{t+1} = Z(\lambda_t) = \begin{cases} 0 & \text{if } M(\lambda_t) < 0 \\ M(\lambda_t) & \text{if } 0 \leq M(\lambda_t) \leq 1 \\ 1 & \text{if } M(\lambda_t) > 1 \end{cases} \tag{5}$$

where

$$M(\lambda_t) = \lambda_t + \lambda_t (1 - \lambda_t) L\gamma \ln(R(\lambda_t)). \tag{6}$$

The parameter γ is introduced as a migration speed. Note that, as M may take values outside the unit interval, the first and last equation of (5) forcefully restrict λ to the unit interval at all times.

Recall that the migration of Puga (1998) is designed for migration between sectors and is justified by the fact that opportunities of migrating between sectors are not available at all times. Rather, they occur according to random law which Puga (1998) takes to be a Poisson process. It also admits the fact that workers may change sector more than once, according to offers made to them. When taken to the context of migration between regions this leads to the admission of workers, who see a job offer in more than one firm of the other region, acting as if they migrate more than once.

Another characteristic of this model, this time inherited from Baldwin (2001), is that if a region is deserted, no workers migrate to it regardless of how high the real wages may be. This seems to be an accepted feature in the literature but not an entirely realistic one. See section 4 below.

3 Chaos: sources and outcomes

The main result of Currie and Kubin (2006), as the title clearly indicates, is the existence of chaotic dynamics for the migration described by (5). Despite the fact that, even considering different time scales, chaotic behaviour does not appear to be a common attitude of workers, it is interesting to understand how this is possible in a model based on well-studied widely-accepted principles. A detailed account of these mechanisms can be found in Garrido-da-Silva (2014), section 4.

A quick comparison of equation (5) with the logistic equation (see Verhulst (1838) and May (1976))

$$x_{t+1} = r x_t (1 - x_t),$$

where $x_t \in [0, 1]$ and $0 \leq r \leq 4$, leads to the identification of the product $L\gamma$ as in the place of r. Then the existence of chaos follows from a period-doubling cascade. Currie and Kubin (2006) prove the existence of chaos with reference to Li and Yorke (1975), after having numerically shown the existence of a period 3 orbit. As with the logistic equation, a low value of $L\gamma$ prevents chaos as is shown in the following

Proposition 3.1. *Given σ and μ, if $L\gamma < \dfrac{2(\sigma - 1)}{\sigma - 1 - \mu\sigma}$ then the dispersion equilibrium does not undergo a period-doubling bifurcation for any value of T.*

The proof proceeds by establishing that $Z'(1/2)$ is never equal to -1 when the condition on $L\gamma$ is satisfied and can be found in Appendix A.

The latter result provides a way to avoid chaos if there is the power to control the total of industrial workers and the migration speed. Assuming, as we are, that these are exogenously defined, chaos may be unavoidable. Considering that the no-black-hole condition, $\sigma - 1 > \mu\sigma$, is in force (see Fujita *et al.* (1999)) the quotient defining the boundary of period-doubling is always positive and becomes very large as μ and σ approach the no-black-hole condition.

Currie and Kubin (2006) report that the period-doubling cascade leading to chaos ends abruptly with a configuration they name *volatile agglomeration* corresponding to concentration of the whole industrial population in a single region. See also Garrido-da-Silva (2014), subsection 3.2 for a detailed description. A striking feature of the volatile agglomeration is that workers concentrate in the region with lowest real wages, thus contradicting the modelling assumption that workers seek the highest real wage.

4 Discrete dynamics

In this section we provide modelling alternatives which comply with the assumptions and natural restrictions of the problem.

4.1 Generic properties

We abandon the assumption that migration opportunities follow a Poisson process and revert to the more commonly used assumption that workers migrate at any time. Thus, the sole assumption is that workers migrate to the region offering the highest real wages.

It seems acceptable that, when the difference between real wages in the two regions is very high more workers are tempted to migrate at a given time. We denote by $m_1 = m_1(R(\lambda_t))$ the percentage of workers who, at the end of time t, migrate from region 2 to region 1. Analogously, $m_2 = m_2(R(\lambda_t))$ denotes the percentage of workers who, at the end of time t, migrate from region 1 to region 2. These functions provide an endogenous description of the migration speed. The assumptions concerning migration are satisfied provided the following holds

C1. $m_1 :]1, +\infty[\to [0, 1]$ and $m_2 :]0, 1[\to [0, 1]$;

C2. $m_1(1) = m_2(1) = 0$ and $m_1(+\infty) = m_2(0) = 1$;

C3. $m_1(R(\lambda_t)) = m_1\left(\frac{\omega_{1,t}}{\omega_{2,t}}\right) = m_2\left(\frac{\omega_{2,t}}{\omega_{1,t}}\right) = m_2(\frac{1}{R(\lambda_t)})$;

C4. m_1 is increasing in $\omega_{1,t}$ and m_2 is increasing in $\omega_{2,t}$.

Note that C1 states that m_1 and m_2 are defined as percentage functions. Condition C2 ensures both that no migration occurs when real wages are equal in both regions, and that there is a block migration of the whole industrial population when total wages are very disparate. The symmetry of migration due to the fixed total population is patent in C3. Condition C4 models the assumption that a bigger difference in real wages leads to more workers migrating at any given time.

The next difference equation characterizes the migration dynamics:

$$\lambda_{t+1} = S(\lambda_t) = \lambda_t + \begin{cases} m_1(R(\lambda_t))\,(1-\lambda_t) & \text{if } R(\lambda_t) \geq 1 \\ \\ -m_2(R(\lambda_t))\lambda_t & \text{if } R(\lambda_t) < 1 \end{cases} \tag{7}$$

Note that the dynamics described by (7) allow for migration to empty regions. Indeed, suppose $\lambda_t = 1$ and $\omega_{1,t} < \omega_{2,t}$, that is, region 2 is empty and offers the highest real wage. Then, the second branch is used to describe the dynamics leading workers out of region 1 and into region 2. The reverse occurs when $\lambda_t = 0$ and $\omega_{1,t} > \omega_{2,t}$.

Proposition 4.1. *Consider the dynamical system described by* $\lambda_{t+1} = S(\lambda_t)$ *as in (7) for all* $t \in \mathbb{N}_0$. *If* $0 \leq \lambda_t \leq 1$ *then* $0 \leq \lambda_{t+1} \leq 1$.

Proof. It suffices to notice that, since both m_1 and m_2 take values in $[0,1]$, the right-hand-side of (7) is such that

$$0 \leq \lambda_t \leq \lambda_t + m_1(R(\lambda_t))(1-\lambda_t) \leq 1 \quad \text{and} \quad 0 \leq \lambda_t - m_2(R(\lambda_t))\lambda_t \leq \lambda_t \leq 1.$$

□

So far, m_1 and m_2 depend on the distribution of industrial workers only implicitly via the real wages. In order to better study the effect of population distribution between the regions, we make this dependence explicit by defining

$$\begin{aligned} m_1 :\ \ ]1,+\infty[\ &\rightarrow\ [0,1] \\ \frac{\omega_{1,t}}{\omega_{2,t}} \ &\mapsto\ \frac{\omega_{1,t}-\omega_{2,t}}{\omega_{1,t}+\omega_{2,t}} = \frac{\frac{\omega_{1,t}}{\omega_{2,t}}-1}{\frac{\omega_{1,t}}{\omega_{2,t}}+1} = \frac{R(\lambda_t)-1}{R(\lambda_t)+1} \end{aligned} \tag{8}$$

and

$$\begin{aligned} m_2 :\ \ ]0,1[\ &\rightarrow\ [0,1] \\ \frac{\omega_{1,t}}{\omega_{2,t}} \ &\mapsto\ \frac{\omega_{2,t}-\omega_{1,t}}{\omega_{1,t}+\omega_{2,t}} = \frac{1-\frac{\omega_{1,t}}{\omega_{2,t}}}{\frac{\omega_{1,t}}{\omega_{2,t}}+1} = \frac{1-R(\lambda_t)}{R(\lambda_t)+1}. \end{aligned} \tag{9}$$

This more particular description of the migration speed recalls the work by Fujita *et al.* (1999), Balwin (2001), and Forsild and Ottaviano (2003) who consider the migration speed proportional to the real wage differential.

The fact that the two regions are identical is patent in the symmetry of the map S. This symmetry is inherited from that of R. The following lemma provides the precise statement. The proof is by direct substitution.

Lemma 4.2. *The map S defined above is such that, for all $t \in [0,1]$, $S(1-\lambda_t) = 1 - S(\lambda_t)$.*

The next subsection establishes the equilibria, and their stability, in a core-periphery model with the dynamics of equation (7).

4.2 Generic configurations and their stability

We begin with a description of the equilibria. As can easily be seen from (8,9), λ^* is a fixed-point of S (hence, an equilibrium) if $R(\lambda^*) = 1$. Other than these, an equilibrium takes place at the boundary, corresponding to a core-periphery configuration. The following proposition is proved in the Appendix A.

Proposition 4.3. *A distribution of industrial workers, λ^* is an equilibrium of the model described in (7) if and only if either $R(\lambda^*) = 1$, $\lambda^* = 0$ or $\lambda^* = 1$.*

Since identically populated regions offer identical real wages, it is then clear that $\lambda = 1/2$ is always an equilibrium. This is the dispersion configuration. A complete study of the dynamics requires the study of the stability of the equilibria. The core-periphery configurations are, as usual, stable when the ratio of real wages favours the populated region. Thus, we have:

Lemma 4.4. *The core-periphery configuration with all industrial workers in region 1, $\lambda^* = 1$, is stable if and only if $R(1) > 1$. The core-periphery configuration with all industrial workers in region 2, $\lambda^* = 0$, is stable if and only if $0 < R(0) < 1$.*

In order to address the remaining configurations, we need to calculate the derivative of S with respect to λ_t. This is given by

$$S'(\lambda_t) = 1 + \begin{cases} \frac{2R'(\lambda_t)}{(R(\lambda_t)+1)^2}(1-\lambda_t) - \frac{R(\lambda_t)-1}{R(\lambda_t)+1} & \text{if } R(\lambda_t) > 1 \\ \\ \frac{2R'(\lambda_t)}{(R(\lambda_t)+1)^2}\lambda_t - \frac{1-R(\lambda_t)}{R(\lambda_t)+1} & \text{if } R(\lambda_t) < 1. \end{cases}$$

The above is not defined for $R(\lambda_t) = 1$ except when $\lambda_t = 1/2$, in which case it is given by

$$S'\left(\frac{1}{2}\right) = 1 + \frac{R'\left(\frac{1}{2}\right)}{4}. \tag{10}$$

Note that, from Lemma 4.2, $S'(\lambda_t) = S'(1-\lambda_t)$. Then, when $0 < R(0) < 1$ or $R(1) > 1$, we can recover the stability of the core-pheripery configurations by demanding that $|S'(1)| = |S'(0)| < 1$.

The stability of the dispersion depends on the magnitude of $S'(1/2)$ which the next lemma shows to be positive for all parameter values. Lemma 4.5 also excludes the existence of period doubling bifurcations from this equilibrium.

Lemma 4.5. *The derivative with respect to λ of S, $S'(1/2, T)$, is positive for all values of T.*

See Appendix A for a proof.

In what concerns stability of symmetric dispersion, we have

Lemma 4.6. *The dispersion configuration, $\lambda = 1/2$, is stable if and only if $R'(1/2) < 0$.*

The proof follows by demanding that S' in (10) be less than 1. Dispersion is then stable when, at $\lambda^* = 1/2$, the ratio of real wages is decreasing in the number of industrial workers in region 1. In this instance, industrial workers in region 2 do not feel tempted to migrate to region 1. By symmetry, the same occurs for industrial workers in region 1 contemplating a move to region 2.

Proposition 4.7. *Asymmetric equilibria, $\lambda^* \in (0, 1)$ such that $\lambda^* \neq 1/2$ and $R(\lambda^*) = 1$, are never stable.*

Proof. Ottaviano (2001), Corollary 2 shows that possible forms of $R(\lambda_t)$ for the Footloose Entrepreneur model. When asymmetric equilibria exist, the same result shows that $R'(\lambda^*) > 0$ for $\lambda^* \neq 1/2$. Hence, the stability conditions of Lemma 4.8 are never satisfied. That the Footloose Entrepreneur model is isomorphic to the core-periphery is shown by Robert-Nicoud (2005, Proposition 2). □

An extension of the standard stability theorem to the case when the derivative does not exist but the limits on the left and right are finite allows for the following

Lemma 4.8. *A configuration with industrial workers in both regions is stable if and only if*

$$min\{-\frac{4}{\lambda^*}, -\frac{4}{1-\lambda^*}\} < R'(\lambda^*) < 0.$$

Proof. Note that since S is differentiable except if $R(\lambda_t) = 1$ and, in this instance, we have the existence of the limits on the right and left, then S is Lipschitz. Auxiliary lemma 4.9 shows[3] that if the modulus of the Lipschitz constant is strictly smaller than 1 then the equilibrium is stable.

[3]We are aware of an alternative proof using Elaydi (2004, p. 182).

Let λ^* be such that $R(\lambda^*) = 1$ and, without loss of generality, assume $\lambda^* < 1 - \lambda^*$. Then

$$S'\left(\lambda^{*+}\right) = \lim_{\lambda \to \lambda^{*+}} \frac{S(\lambda) - S(\lambda^*)}{\lambda - \lambda^*} = 1 + \frac{1-\lambda^*}{2} R'(\lambda^*)$$

and

$$S'\left(\lambda^{*-}\right) = \lim_{\lambda \to \lambda^{*-}} \frac{S(\lambda) - S(\lambda^*)}{\lambda - \lambda^*} = 1 + \frac{\lambda^*}{2} R'(\lambda^*).$$

Also,

$$\begin{aligned} \left|S'\left(\lambda^{*+}\right)\right| < 1 \quad &\Leftrightarrow \quad \left|1 + \frac{\lambda^*}{2} R'(\lambda^*)\right| < 1 \Leftrightarrow \\ &\Leftrightarrow \quad R'(\lambda^*) \in \left(-\frac{4}{\lambda^*}, 0\right). \end{aligned}$$

Analogously, we have

$$\left|S'\left(\lambda^{*-}\right)\right| < 1 \quad \Leftrightarrow \quad R'(\lambda^*) \in \left(-\frac{4}{1-\lambda^*}, 0\right).$$

This is enough to guarantee that S is Lipschitz with constant of modulus strictly less than 1, finishing the proof. □

Lemma 4.9. *Let f be Lipschitz with constant $|K| < 1$ and x^* be a fixed-point of f. Then x^* is locally asymptotically stable for $x_{t+1} = f(x_t)$.*

Proof. We have

$$\begin{aligned} |f^n(x) - f^n(x^*)| = |f^n(x) - x^*| \quad &\leq \quad K|f^{n-1}(x) - f^{n-1}(x^*)| \leq \\ \leq \ldots &\leq \quad K^n |x - x^*|. \end{aligned}$$

Taking limits of both sides, since $|K| < 1$, we obtain

$$|f^n(x) - f^n(x^*)| \to 0$$

as $n \to +\infty$. □

4.3 New dynamics

In order to describe the dynamics for the new model, we first observe that it follows from the calculations in Lemmas 4.4 and 4.6 that it is possible to describe stable regions for dispersion and concentration with respect to T.

Lemma 4.10. *The dispersion configuration, $\lambda = 1/2$, is stable for $T > T_B$ and either core-periphery configuration is stable if $T < T_S$ where*

$$T_B = \left(\frac{(\sigma - 1 + \mu\sigma)(1 + \mu)}{(\sigma - 1 - \mu\sigma)(1 - \mu)} \right)^{\frac{1}{\sigma - 1}}$$

and T_S is given as a solution of

$$\frac{1 - \mu}{2} T^{\sigma - 1 - \mu\sigma} + \frac{1 + \mu}{2} T^{1 - \sigma - \mu\sigma} = 1.$$

The proof is given in Appendix A. Robert-Nicoud (2005, Proposition 5) establishes that $T_B < T_S$ and therefore, symmetric dispersion and agglomeration coexist as stable configurations.

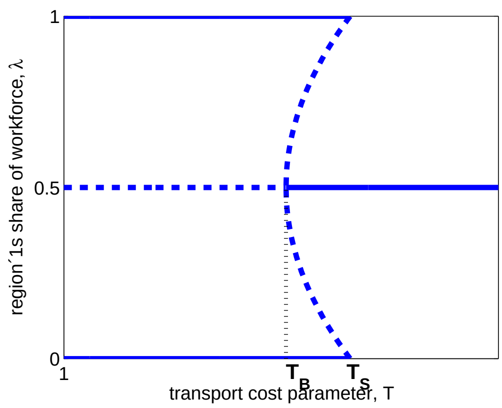
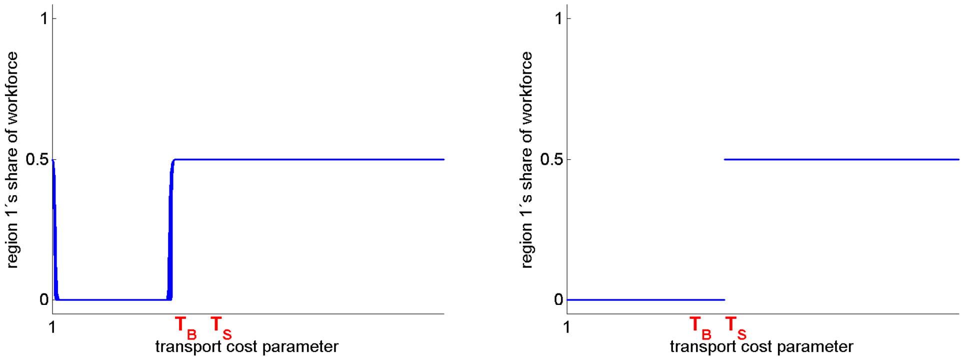


Figure 1: Bifurcation diagram for $\mu = 0.4$ and $\sigma = 5$. Core-periphery equilibria exist for $T \leq T_S$. Symmetric dispersion is a steady-state for all parameter values, stable for $T \geq T_B$. For parameter values $T_B < T < T_S$, two asymmetric dispersion configurations appear. These are always unstable.

We are now able to draw the bifurcation diagram in Figure 1. This shows that the discrete-time behaviour of the core-periphery model is very similar to that of the continuous-time model.

In the remaining of this subsection, we describe and simulate the dynamics for $\mu = 0.4$ and $\sigma = 5$. This allows for a more detailed description of the convergence properties of solutions.

Figure 2 shows, for different values of T, the equilibrium to which trajectories starting from two different initial conditions converge. The two different initial conditions are $\lambda_0 = 0.499$ and $\lambda_0 = 0.001$. Either is close to a different equilibrium so that the coexistence of dispersion and agglomeration for $T \in (T_B, T_S)$ is illustrated. In fact, in Figure 2a, where the initial



(a) $\lambda_0 = 0.499$ (b) $\lambda_0 = 0.001$

Figure 2: For $\mu = 0.4$ and $\sigma = 5$ the figure shows the convergence from the given initial condition: initial condition is (a) $\lambda_0 = 0.499$ and (b) $\lambda_0 = 0.001$.

condition is close to dispersion, already for T close to T_B the trajectory ends at the dispersion configuration. On the other hand, Figure 2b, shows that trajectories starting near agglomeration do not converge to dispersion until T reaches T_S. We point out the difference in the line thickeness patent in cases (a) and (b). This illustrates different speeds characteristcs of the convergence process as described next.

Figure 3 shows, when $\lambda_0 = 0.499$, the speed of convergence, through the number of iterations, for values of T slightly higher that $T = 1$ and slightly lower than $T = T_B$. We observe that the speed of convergence is slower for values of T closer to either $T = 1$ or $T = T_B$.

5 Concluding remarks

We propose a new discrete-time model for the classis NEG problem of two regions. This model respects the dynamics previously established for continuous-time. The absence of chaotic behaviour makes the present model highly suitable for numerical study/simulations and it opens the path for generalizations to a higher number of regions.

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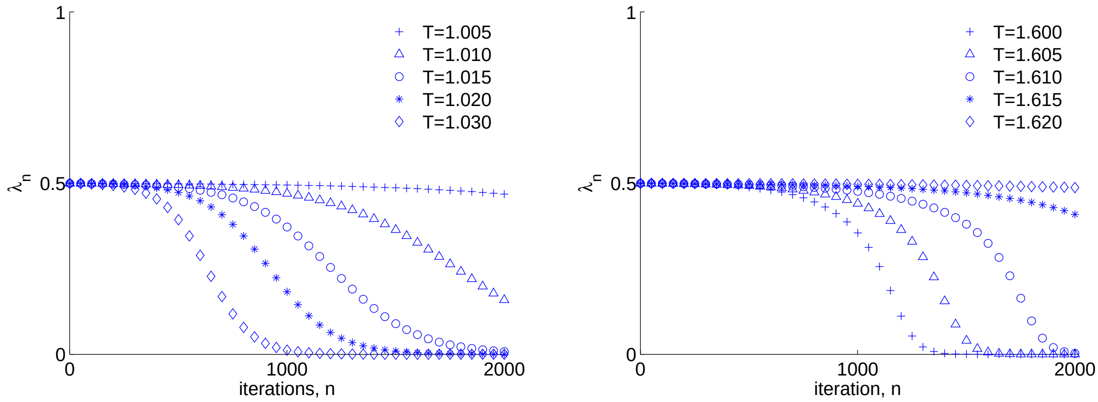

(a) (b)

Figure 3: For $\mu = 0.4$, $\sigma = 5$ and $\lambda_0 = 0.499$ the figure shows the speed of convergence from the given initial condition: (a) T moves away for 1 and (b) T approaches T_B on the left.

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Appendix A: Proofs

This appendix is devoted to the lengthier and/or more technical proofs. We reproduce the statement of each result for ease of reference.

Proposition 3.1. *Given σ and μ, if $L\gamma < \dfrac{2(\sigma-1)}{\sigma-1-\mu\sigma}$ then the dispersion equilibrium does not undergo a period-doubling bifurcation for any value of T.*

Proof. Let $0 < \mu < 1$, $\sigma > 1$ and $0 < L\gamma < \dfrac{2(\sigma-1}{\sigma-1-\mu\sigma}$. A necessary condition for a period-doubling bifurcation is $Z'\left(\dfrac{1}{2}, T\right) = -1$. Straighforward calculations show that this is equivalent to

$$L\gamma = 2\frac{\sigma-1}{1-T^{\sigma-1}}\frac{(1-T^{\sigma-1})\left[\mu\left(1+T^{\sigma-1}\right)+1-T^{\sigma-1}\right]+4\sigma T^{\sigma-1}}{\mu\left(2\sigma-1\right)\left(1+T^{\sigma-1}\right)+\left(\sigma-1+\mu^2\sigma\right)\left(1-T^{\sigma-1}\right)}.$$

The above equality describes the product $L\gamma$ as a function of T. It is easy to see that

$$\lim_{T\to\infty} L\gamma = \frac{2(\sigma-1)}{\sigma-1-\mu\sigma} \text{ and } \lim_{T\to T_B} L\gamma = +\infty,$$

thus showing that the necessary condition is not satisfied in the specified parameter range. ☐

Proposition 4.3. *A distribution of industrial workers, λ^* is an equilibrium of the model described in (7) if and only if either $R(\lambda^*) = 1$, $\lambda^* = 0$ or $\lambda^* = 1$.*

Proof. An equilibrium, λ^*, satisfies $S(\lambda^*) = \lambda^*$. If $R(\lambda_t) \geq 1$ then

$$\lambda^* + \frac{R(\lambda^*) - 1}{R(\lambda^*) + 1}(1 - \lambda^*) = \lambda^* \Leftrightarrow R(\lambda^*) = 1 \wedge \lambda^* = 1.$$

Analogously, if $0 < R(\lambda_t) \leq 1$, we obtain $R(\lambda^*) = 1 \wedge \lambda^* = 0$. ☐

Lemma 4.5. *The derivative with respect to λ of S, $S'(1/2, T)$, is positive for all values of T.*

Proof. Considering the possible forms for the ratio of real wages shown by Ottaviano (2001, p. 59), it becomes clear that

$$\min_{T>1} R(0,T) = \max_{T>1} R'\left(\frac{1}{2}, T\right) = \max_{T>1} \left\{1 + \frac{R'\left(\frac{1}{2}, T\right)}{4}\right\} = \max_{T>1} S'\left(\frac{1}{2}, T\right).$$

We thus look at the functions of T defined by

$$R(0,T) = \left[\frac{1 - \mu + (1+\mu)\, T^{2(1-\sigma)}}{2T^{1-\sigma}}\right]^{\frac{1}{\sigma}} T^{-\mu},$$

$$S'\left(\tfrac{1}{2}, T\right) = 1 - \frac{1 - T^{\sigma-1}}{\sigma - 1} \frac{\mu\,(2\sigma - 1)\,(1 + T^{\sigma-1}) + (\sigma - 1 + \mu^2\sigma)\,(1 - T^{\sigma-1})}{(1 - T^{\sigma-1})\,[\mu\,(1 + T^{\sigma-1}) + 1 - T^{\sigma-1}] + 4\sigma T^{\sigma-1}}.$$

Sufficient conditions for a local interior minimum, attained at a value $\hat{T} > 1$, are

(i) $\dfrac{\partial R}{\partial T}\left(0, \hat{T}\right) = 0;$

(ii) $\dfrac{\partial^2 R}{\partial T^2}\left(0, \hat{T}\right) > 0.$

From (i), we have

$$\begin{aligned} \frac{\partial R}{\partial T}(0,T) = {} & \frac{1}{2\sigma}\left[\frac{1 - \mu + (1+\mu)\, T^{2(1-\sigma)}}{2T^{1-\sigma}}\right]^{\frac{1}{\sigma}-1} T^{\sigma-\mu-2} \\ & \times \left[(1+\mu)\,(1 - \sigma - \sigma\mu)\, T^{2(1-\sigma)} - (1-\mu)\,(1 - \sigma + \sigma\mu)\right]. \end{aligned}$$

Since $\frac{1}{2\sigma}\left[\frac{1-\mu+(1+\mu)T^{2(1-\sigma)}}{2T^{1-\sigma}}\right]^{\frac{1}{\sigma}-1} T^{\sigma-\mu-2} \neq 0$, for any σ, $T>1$ and $0<\mu<1$, it follows that

$$\frac{\partial R}{\partial T}(0,T)=0 \Leftrightarrow \quad T\equiv\hat{T}=\left[\frac{(1+\mu)(\sigma-1+\sigma\mu)}{(1-\mu)(\sigma-1-\sigma\mu)}\right]^{\frac{1}{2(\sigma-1)}}.$$

As for (ii), we have

$$\begin{aligned}\frac{\partial^2 R}{\partial T^2}(0,T)= & \frac{1}{4\sigma^2}\left[\frac{1-\mu+(1+\mu)T^{2(1-\sigma)}}{2T^{1-\sigma}}\right]^{\frac{1}{\sigma}-2} T^{2\sigma-\mu-4} \\ \times & \left\{(1+\mu)^2(\sigma-1+\sigma\mu)(2\sigma-1+\sigma\mu)T^{4(1-\sigma)}\right. \\ & +2(1+\mu)(1-\mu)\left[(1-\sigma)^2(2\sigma-1)+\sigma^2\mu(1+\mu)\right]T^{2(1-\sigma)} \\ & \left.-(1-\mu)^2(\sigma-1-\sigma\mu)(1+\sigma\mu)\right\}\end{aligned}$$

Clearly, $\frac{1}{4\sigma^2}\left[\frac{1-\mu+(1+\mu)T^{2(1-\sigma)}}{2T^{1-\sigma}}\right]^{\frac{1}{\sigma}-2} T^{2\sigma-\mu-4}>0$, for all σ, $T>1$ e $0<\mu<1$. Therefore, in order to prove that $\frac{\partial^2 R}{\partial T^2}\left(0,\hat{T}\right)>0$, it suffices to check the sign of the term in brackets. Replacing $\hat{T}$ we obtain

$$\begin{aligned}&(1+\mu)^2(\sigma-1+\sigma\mu)(2\sigma-1+\sigma\mu)\hat{T}^{4(1-\sigma)}+2(1+\mu)(1-\mu)\\ &\times\left[(1-\sigma)^2(2\sigma-1)+\sigma^2\mu(1+\mu)\right]\hat{T}^{2(1-\sigma)}-(1-\mu)^2(\sigma-1-\sigma\mu)(1+\sigma\mu)=\end{aligned}$$

$$=\quad\frac{(1-\mu)^2(\sigma-1-\sigma\mu)}{\sigma-1+\sigma\mu}\times 4\sigma(\sigma-1)^2>0,$$

since $\sigma>1$, $0<\mu<1$ and the *no-black-hole* condition $\sigma-1>\sigma\mu$ hold.

We have thus proved that $R\left(0,\hat{T}\right)$ and $S'\left(\frac{1}{2},\hat{T}\right)$ are, respectively, a local minimum of $R(0,T)$ and a local maximum of $S'\left(\frac{1}{2},T\right)$, with

$$\hat{T}=\left[\frac{(1+\mu)(\sigma-1+\sigma\mu)}{(1-\mu)(\sigma-1-\sigma\mu)}\right]^{\frac{1}{2(\sigma-1)}}.$$

Note that the domain of $S'\left(\frac{1}{2},T\right)$ is the interval $(1,+\infty)$. Also

$$\begin{aligned}\lim_{T\longrightarrow 1^+} S'\left(\frac{1}{2},T\right) &=1<S'\left(\tfrac{1}{2},\hat{T}\right),\\ \lim_{T\longrightarrow +\infty} S'\left(\frac{1}{2},T\right) &=1-\frac{\sigma-1-\sigma\mu}{\sigma-1}<S'\left(\frac{1}{2},\hat{T}\right),\end{aligned}$$

meaning that, in fact,

- $S'\left(\frac{1}{2},\hat{T}\right)$ is the unique absolute maximum of $S'\left(\frac{1}{2},T\right)$;
- $S'\left(\frac{1}{2},(1,+\infty)\right)=\left(1-\frac{\sigma-1-\mu\sigma}{\sigma-1},S'\left(\frac{1}{2},\hat{T}\right)\right]\subset\left(0,S'\left(\frac{1}{2},\hat{T}\right)\right]$.

Hence, $S'\left(\frac{1}{2},T\right)>0$, for all $T>1$.

□

Lemma 4.10. *The dispersion configuration, $\lambda = 1/2$, is stable for $T > T_B$ and either core-periphery configuration is stable if $T < T_S$ where*

$$T_B=\left(\frac{(\sigma-1+\mu\sigma)(1+\mu)}{(\sigma-1-\mu\sigma)(1-\mu)}\right)^{\frac{1}{\sigma-1}}$$

and T_S is given as a solution of

$$\frac{1-\mu}{2}T^{\sigma-1-\mu\sigma}+\frac{1+\mu}{2}T^{1-\sigma-\mu\sigma}=1.$$

Proof. Dispersion is stable provided $|S'(1/2,T)|<1$ which corresponds to

$$R'\left(\frac{1}{2},T\right)<0 \quad\Leftrightarrow$$
$$\Leftrightarrow\quad T>T_B=\left(\frac{(\sigma-1+\mu\sigma)(1+\mu)}{(\sigma-1-\mu\sigma)(1-\mu)}\right)^{\frac{1}{\sigma-1}}.$$

Note that, given the no-black-hole condition, all quantities in brackets are positive.

Stability of $\lambda = 0$ is ensured by $0 < R(0) < 1$, which when solved for equality provides

$$R(0,T)=1 \quad\Leftrightarrow$$
$$\Leftrightarrow\frac{1-\mu}{2}T_S^{\sigma-1-\mu\sigma}+\frac{1+\mu}{2}T_S^{1-\sigma-\mu\sigma}=1.$$

□

Appendix B: Particular alternatives

This appendix contains alternatives for the functions m_1 and M_2 which still comply with conditions $C1 - C4$ of the new model. They may be used to impose or model different weights in the migration process.

The Euclidean norm: The Euclidean norm is

$$||(x_1,\dots,x_n)||_2 = \sqrt{x_1^2+\dots+x_n^2}.$$

We choose for m_1 and m_2 the following:

$$\begin{array}{rccl} m_1: &]1,+\infty[& \to & [0,1] \\ & \dfrac{\omega_{1,t}}{\omega_{2,t}} & \mapsto & \dfrac{\omega_{1,t}-\omega_{2,t}}{\sqrt{\omega_{1,t}^2+\omega_{2,t}^2}} = \dfrac{\frac{\omega_{1,t}}{\omega_{2,t}}-1}{\sqrt{\left(\frac{\omega_{1,t}}{\omega_{2,t}}\right)^2+1}}, \end{array}$$

$$\begin{array}{rccl} m_2: &]0,1[& \to & [0,1] \\ & \dfrac{\omega_{1,t}}{\omega_{2,t}} & \mapsto & \dfrac{\omega_{2,t}-\omega_{1,t}}{\sqrt{\omega_{1,t}^2+\omega_{2,t}^2}} = \dfrac{1-\frac{\omega_{1,t}}{\omega_{2,t}}}{\sqrt{\left(\frac{\omega_{1,t}}{\omega_{2,t}}\right)^2+1}}. \end{array}$$

The maximum norm: The maximum norm is

$$||(x_1,\dots,x_n)||_\infty = \max_{i\in\{1,2,\dots,n\}} |x_i|.$$

The functions m_1 and m_2 are then constructed as follows:

$$\begin{array}{rccl} m_1: &]1,+\infty[& \to & [0,1] \\ & \dfrac{\omega_{1,t}}{\omega_{2,t}} & \mapsto & \dfrac{\omega_{1,t}-\omega_{2,t}}{\max\{\omega_{1,t},\omega_{2,t}\}} = 1-\dfrac{1}{\frac{\omega_{1,t}}{\omega_{2,t}}}, \end{array}$$

$$\begin{array}{rccl} m_2: &]0,1[& \to & [0,1] \\ & \dfrac{\omega_{1,t}}{\omega_{2,t}} & \mapsto & \dfrac{\omega_{2,t}-\omega_{1,t}}{\max\{\omega_{1,t},\omega_{2,t}\}} = 1-\dfrac{\omega_{1,t}}{\omega_{2,t}}. \end{array}$$

The logistic function: This function belongs to the wider class of bounded one-variable functions with positive derivative at all points. It is canonically described as

$$f(x) = \frac{1}{1+\exp(-x)}$$

and induces

$$\begin{array}{rccl} m_1: &]1,+\infty[& \to & [0,1] \\ & \dfrac{\omega_{1,t}}{\omega_{2,t}} & \mapsto & 1 - \dfrac{2}{1+\exp\left(\frac{\omega_{1,t}}{\omega_{2,t}} - 1\right)}, \end{array}$$

$$\begin{array}{rccl} m_2: &]0,1[& \to & [0,1] \\ & \dfrac{\omega_{1,t}}{\omega_{2,t}} & \mapsto & 1 - \dfrac{2}{1+\exp\left(\frac{1}{\frac{\omega_{1,t}}{\omega_{2,t}}} - 1\right)}. \end{array}$$

The arc-tangent function: Again we look at a bounded function with positive derivative at all points, this time with target restricted to the interval $[-\pi/2, \pi/2]$, such that $f(x) = \arctan(x)$. This gives rise to

$$\begin{array}{rccl} m_1: &]1,+\infty[& \to & [0,1] \\ & \dfrac{\omega_{1,t}}{\omega_{2,t}} & \mapsto & \dfrac{2}{\pi}\arctan\left(\dfrac{\omega_{1,t}}{\omega_{2,t}} - 1\right), \end{array}$$

$$\begin{array}{rccl} m_2: &]0,1[& \to & [0,1] \\ & \dfrac{\omega_{1,t}}{\omega_{2,t}} & \mapsto & \dfrac{2}{\pi}\arctan\left(\dfrac{1}{\frac{\omega_{1,t}}{\omega_{2,t}}} - 1\right). \end{array}$$