Price competition between verti-zontally differentiated platforms

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Abstract. We study a two-sided market duopoly in which the differentiated platforms compete in prices. Platforms are differentiated both vertically and horizontally (verti-zontal differentiation). We develop a model that is a synthesis of the two-sided market model of Armstrong (2006) with Neven and Thisse (1990), who study price competition between vertically and horizontally conventional firms. Considering a symmetric baseline model, in which price discrimination between sides cannot take place, we find that equilibrium outcomes depend on the strength of the inter-group network effects vis-à-vis the magnitude of the intrinsic quality differences between the platforms. Moreover we find that, under horizontal dominance, the profit of the low-quality platform may decrease (or increase) as the quality of the better product improves and, under vertical dominance, the profit of the high-quality platform is increasing with an increment on the intensity of the network effect.

Keywords: Two-sided markets, Horizontal differentiation, Vertical differentiation.

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1 Introduction

In two-sided markets, firms (platforms) are intermediaries between two categories of agents, such that at least one category of agents generates inter-group network effects over the other group of agents in the market. Accordingly, in these markets, the value of participating in a certain platform (for a certain category of agents) depends not only on its intrinsic characteristics but also on the number of agents (of a different category) participating in the same platform.

Examples of two-sided platforms include: (*i*) media outlets, which allow for the interaction between advertisers and readers; (*ii*) clubs, which allow for the interaction between men and women; (*iii*) e-commerce platforms, which allow for the interaction between buyers and sellers, and so on.

In the context of the examples above, real life reveals every day plethoric situations in which platforms are simultaneously vertically and horizontally differentiated [4]. In the case of clubs, we observe that they are horizontally differentiated (including the type of music they play or their environment) as well as vertically differentiated (quality of the air, quality of the food, parking space). The same occurs in the case of e-commerce platforms. Similarly, in the case of e-commerce platforms, they often differ along horizontal dimensions (such as their aesthetics, the type of products they sell) as well as vertical dimensions (like their usability, delivery times and advertising intensity).

In this paper, we aim at investigating the nature of price competition between platforms that are both horizontally and vertically differentiated. Using the terminology proposed by Di Comite et al. (2011), in the context of monopolistic competition, we may say that our objective is to analyze price competition between verti-zontally differentiated platforms.

Following the seminal works of Rochet and Tirole (2003) and Armstrong (2006), a vast literature has analyzed the strategic aspects of competition between two-sided plat-

[4] See Gabszewicz and Resende (2012) for further details concerning horizontal and vertical differentiation in the press industry.

forms. In particular, following Armstrong (2006), a considerable number of works has been devoted to the analysis of strategic interaction between horizontally differentiated platforms (see, for example, Zacharias and Serfes, 2012; Gabszewicz et al., 2013).

In this paper, we combine two strands of literature: the literature on price competition under horizontal and vertical differentiation and the literature on two-sided markets. Concerning the first line of literature, Economides (1989) and Neven and Thisse (1990) both analyze a two-dimensional vertical and horizontal differentiation model in which firms compete on quality, variety and price.

Economides (1989) assumes that the variety (horizontal) choice takes place before the quality (vertical) choice. Considering that marginal costs are increasing in quality levels, he concludes that in equilibrium, we observe maximum variety differentiation and minimum quality differentiation.

Neven and Thisse (1990) consider a three-stage game, in which firms first choose their quality, afterwards their location, competing in prices at the end of the game. Normalizing the marginal costs to zero, the authors find a product equilibrium that exhibits maximum differentiation on one dimension and minimum differentiation on the other. The maximally differentiated dimension can either be the quality-or variety dimension.

More recently, other authors have addressed the issue of price competition in conventional markets with horizontal or/and vertical differentiation (see, for example, Levin, Peck and Ye (2009), Gabsewicz and Resende (2012), Gabszewicz and Wauthy (2014). The present paper contributes to this literature by investigating how the nature of price competition between verti-zontally differentiated firms can be affected by inter-group network effects.

Our paper also contributes to the literature on two-sided markets. Most of the literature on this field has focused on the waterbed effects and tipping prevention (see for example, Rochet and Tirole (2003); Armstrong (2006). A considerable literature has also addressed the issue of price competition between horizontally differentiated platforms (see for example, Zacharias and Serfes (2012); Gabszewicz et al. (2013). Using the Hotelling framework proposed by Armstrong (2006), these works have studied pricing competition between horizontally differentiated platforms. When agents are allowed to participate in only one platform, these papers (including Armstrong, 2006) show that the existence of inter-group network effects often intensifies price competition between the platforms. To the best of our knowledge, Gabszewicz et al. (2013) is the first paper that simultaneously combines horizontal and vertical differentiation in two-sided markets. However, the authors do not analyze price competition between the two-sided platforms. We contribute to this literature by analyzing how the impact of inter-group network effects on equilibrium outcomes depends on the interplay of vertical and horizontal differentiation.

In our model, we bring together the verti-zontal differentiation set-up developed by Neven and Thisse (1990) and the two-sided market framework proposed by Armstrong (2006). Platforms are assumed to be both horizontally and vertically differentiated. On the horizontal dimension, we consider a Hotelling framework in line with the two previous papers. In line with Armstrong (2006), we consider that firms are exogenously located at the extremes of the Hotelling line. As in Neven and Thisse (1990) platforms are also vertically differentiated and consumers have heterogeneous willingness to pay for quality. In addition, we allow for inter-group network effects, with the objective of studying how they affect pricing strategies of verti-zontally differentiated firms.

In the baseline model, we do not allow for price discrimination between sides (as an illustrative example, the reader may consider of a club where men and women pay the same entry fee, or a e-commerce platform, where sellers and buyers are required to pay the same access fee in order to participate in the platform). The paper proposes a two-stage game with the following structure. In the first stage, platforms set their access prices and afterwards consumers decide which platform they are willing to join.

Our equilibrium analysis shows that the high-quality platform sets a higher access price, serves a larger market share and earns a higher profit than the low-quality platform, in line with standard vertical differentiation literature (see, for example, Tirole (2003). We show that equilibrium outcomes depend on the strength of the inter-group network effects *vis-à-vis* the magnitude of the intrinsic quality differences between the platforms.

Traditional literature that combines simultaneously horizontal and vertical differentiation argues that revenues of both firms increase as the quality of the better product improves (see Shaked and Sutton, 1982).

Our finding establishes that this argument may be disrupt under horizontal dominance since we find that, with the presence of network effects and under a subset of pure horizontal dominance, as the quality of the product sold by the high-quality platform increases more relatively to the product sold by its rival, the revenue of the low-quality platform decreases. However, when the quality gap between platforms is too high (but still in an horizontal dominance environment) follows that an increment on the quality gap is also profit enhancing for the low-quality platform.

Thus, this evidence suggests that when the high-quality platform puts an excessive effort on increasing the vertical differentiation between platforms in a horizontal dominance environment constitutes a strategy that benefits the low-quality platform.

Moreover we find that, under vertical dominance, the profit of the high-quality platform is increasing with an increment on the intensity of the network effect, which constitutes a distinct finding relatively to Armstrong (2006). Here the intuition relies on the fact that when the members of both sides of the market attribute too much value to quality rather than variety, the high-quality platform benefits with a marginally increment on the intensity of the network effect.

The rest of the paper is organized as follows. In the next section 2 we present the model. Section 3 analyzes the model and section 4 characterizes the equilibrium outcomes. Section 5 performs a comparative statics. Finally section 6 concludes. The proofs of the Propositions and Lemmas are relegated to the Appendix.

2 The model

Consider a two-sided market with two platforms, $i \in \{A, B\}$, and two sides, $j \in \{1, 2\}$. The platforms operate with zero marginal costs[5] and they are both horizontally and vertically differentiated. As in Neven and Thisse (1990), we assume that the platforms differ in two characteristics: (i) their location on the Hotelling line (horizontal differentiation), and (ii) their intrinsic quality (vertical differentiation).

In particular, platform A is located at the left extreme of the Hotelling line $\left(x^A = 0\right)$, whereas platform B is located at the right extreme $\left(x^B = 1\right)$. The intrinsic quality of platform A is denoted by q^A, whereas the intrinsic quality of platform B is q^B, with $q^B > q^A$. Then, platform A is the low-quality platform while platform B is the high-quality platform.

In each side of the market, there is a unit mass of heterogeneous consumers. They differ on their location, x, as well as on their quality valuation, y. Accordingly, in side j, each consumer is identified by the pair (x, y). As in Neven in Thisse (1990), we assume that in each side of the market consumers are uniformly distributed over the unit square $[0, 1] \times [0, 1]$ and we normalize their transportation costs to 1.

We depart from Neven and Thisse (1990) by introducing the possibility of positive inter-group externalities. As in Serfes and Zacarias (2012), we assume that the intensity of inter-group externalities is the same on both sides of the market, being measured by the parameter $\alpha \geq 0$.

In the market side $j \in \{1, 2\}$, a consumer type $(x, y) \in [0, 1] \times [0, 1]$ obtains the following utility from participating in platform A:

$$u_j^A(x, y) = v + yq^A + \alpha D_k^A - p^A - x,$$

where $v \in \mathbb{R}_+$ is sufficiently high for the market to be covered in equilibrium, D_k^A is the demand of platform A on the other side ($k \neq j$) and p^A is the access price charged by

[5] Our results would remain unchanged under constant and symmetric marginal costs. A similar assumption has been adopted by Neven and Thisse (1990), among others.

platform A, which applies to both sides of the market (price discrimination between sides is not allowed).[6]

Analogously, when participating in platform B, a side-j consumer of type (x, y) obtains the following utility:

$$u_j^B(x,y) = v + yq^B + \alpha D_k^B - p^B - (1-x),$$

where D_k^B is the demand of platform B on the other side ($k \neq j$) and p^B is the access price charged by platform B to both sides of the market.

Notice that the intensity of the network effect, captured by α, influences both variety (the degree of horizontal differentiation) and quality (the degree of vertical differentiation).

In order to study equilibrium outcomes when platforms are simultaneously horizontally and vertically differentiated, we consider a game with the following timing: in the first stage, platforms simultaneously set access prices for both sides; in the second stage, consumers in each side simultaneously decide which platform to join. The game is solved by backward induction.

We start by investigating the specification of demand resulting from the second stage. Afterwards, we analyze platforms' equilibrium price choices, in the first stage. Through the paper, we assume that the inter-group externality is relatively weak in order to avoid market tipping.

Assumption 1 (Weak inter-group externality) *The inter-group externality is relatively weak:* $\alpha < \frac{1}{2}$.

3 Demand and profits

In side j, the consumer type $(\widetilde{x}_j(y), y)$ for whom

$$u_j^A(\widetilde{x}_j(y), y) = u_j^B(\widetilde{x}_j(y), y)$$

[6] If price discrimination was allowed, we would expect the symmetry of the model to induce no discrimination in equilibrium.

is indifferent between participating in platform A or platform B. Solving the previous equation for $\widetilde{x}_j(y)$, we obtain that, for a given y, the indifferent consumer in side j is located at:

$$\widetilde{x}_j(y) = \tfrac{1}{2} + \tfrac{p^B - p^A}{2} + \tfrac{\alpha(D_k^A - D_k^B)}{2} - \tfrac{q}{2}y, \tag{1}$$

where q represents the quality gap, i.e. $q \equiv q^B - q^A$.

Taking into consideration that the market is fully covered, $D_k^B = 1 - D_k^A$ and that consumers formulate self-fulfilled expectations about the demand on the other side of the market, with $D_k^A = \int_0^1 \widetilde{x}_k(y)\,dy$ for $\widetilde{x}(y_j) \in [0,1]$,[7] equation (1) can be re-written as follows:

$$\widetilde{x}_j(y) = \frac{1}{2} + \frac{p^B - p^A}{2} + \frac{\alpha(2D_k^A - 1)}{2} - \frac{q}{2}y, \tag{2}$$

or equivalently:

$$\widetilde{y}_j(x) = \frac{1}{q} + \frac{p^B - p^A}{q} + \frac{\alpha(2D_k^A - 1)}{q} - \frac{2}{q}x. \tag{3}$$

The previous equations show that for a given vector of prices $\left(p^A, p^B\right)$ the position of the indifferent consumer $\widetilde{y}_j$ evolves linearly and negatively with x, as $\frac{\partial \widetilde{y}_j(x)}{\partial x} = -\frac{2}{q} < 0$.

Consider two types of consumers: (x, y) and (x', y'), with $x' < x$, meaning that the second consumer is located closer to firm A. The two consumers are only both indifferent between participating in platform A and B if the second consumer has a higher willingness to pay for quality than the first one i.e. $y' > y$.

Note that, if, for a given x, we have $\widetilde{y}_j(x) \in [0,1]$, for such x, side j consumers located at $y_j \in [0, \widetilde{y}_j(x)]$ participate in platform A, whereas those located at $y_j \in (\widetilde{y}_j(x), 1]$ participate in platform B. In other words, consumers above the line (3) participate in platform B, whereas those below that line prefer to participate in platform A. If we have $\widetilde{y}(x_j) < 0 \forall x \in [0,1]$, resp. $\widetilde{y}(x_j) > 1 \forall x \in [0,1]$, then, all consumers in side j buy from platform B, resp. platform A.

The figure 1 below illustrates the choice of consumers between the two platforms for

[7] This formulation means that an agent of a side j $(j = 1, 2)$ can interact with an agent of an opposite side k ($k = 1$ when $j = 2$ or $k = 2$ when $j = 1$), for any given variety x and quality y within the square $[0,1] \times [0,1]$.

different pairs of prices $\left(p^A, p^B\right)$.

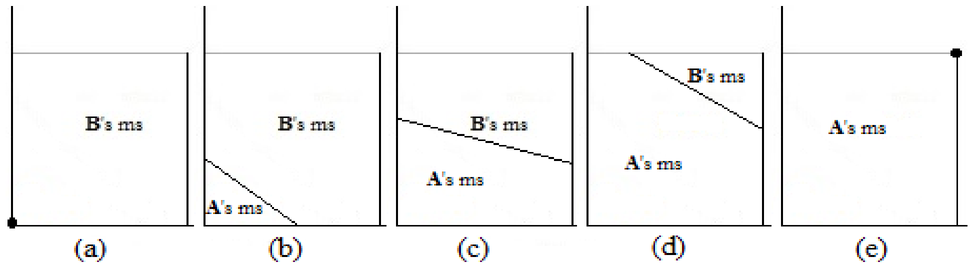

Figure 1: Consumers choices between platforms under vertical dominance

Considering the position of the indifferent consumer line, it is now possible to obtain the analytical expressions of the demands of platform A and platform B, in side j, as a function of firms´ access prices, p^A and p^B.

As in Neven in Thisse (1990), in order to obtain these analytical expressions we first need to distinguish two cases according to the relative importance that consumers attach to vertical differentiation *vis-à-vis* horizontal differentiation.

Definition 1 ***(Horizontal Dominance versus Vertical Dominance)***

(i) *Horizontal dominance corresponds to an environment where the intrinsic quality gap between the platforms, q, is relatively low, in particular, $q < 2$.*

(ii) *Instead, when the intrinsic quality gap is sufficiently large, $q > 2$, Vertical dominance prevails.*

Proof. *As in Neven and Thisse (1990), horizontal (resp. vertical) dominance arises when $\left|\frac{\partial \widetilde{y}(x)}{\partial x}\right| > (<) 1$, or equivalently, $q < (>) 2$.* ■

In order to better understand the intuition and the need for the distinction pointed out in Definition 1, consider the two cases of figure 2.

For the values of the parameters considered in figure 18(b), we have $\left|\frac{\partial \widetilde{y}(x)}{\partial x}\right| < 1$, or equivalently $q > 2$. As in this case, the intrinsic quality gap between the two firms is relatively large, Neven and Thisse (1990) introduce the concept of vertical dominance to denote the domain of parameters for which $\left|\frac{\partial \widetilde{y}(x)}{\partial x}\right| < 1$.

However, when $q < 2$, or equivalently, $\left|\frac{\partial \widetilde{y}(x)}{\partial x}\right| > 1$, the intrinsic quality gap between the firms is relatively low and horizontal dominance prevails. In that case, for the same price vector $\left(p^A, p^B\right)$ we would obtain a different demand configuration as illustrated in the figure $2(a)$.

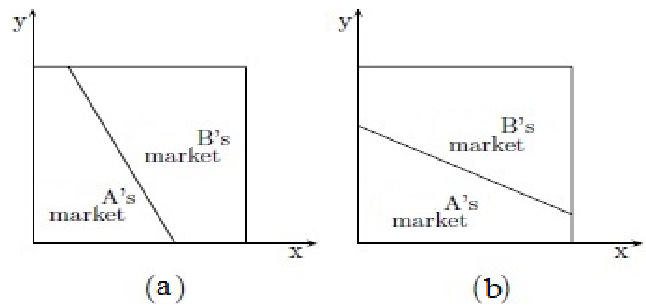

Figure 2: Horizontal versus vertical dominance

In light of this, for each side of the market j, we now define the analytical expressions of the platforms' demands as a function of their access prices $\left(p^A, p^B\right)$, under horizontal and vertical dominance[8].

We denote by $D_j^{i|VD}\left(p^A, p^B\right)$ the demand of platform i in side j, under vertical dominance, with:

[8]See Appendix A in section 7.1 for further details on the computation of the analytical expression of the demand functions of each platform, under horizontal and vertical dominance.

$D_j^{A|VD}\left(p^A, p^B\right) =$

$$\begin{cases} 0 & if \quad p^A - p^B > 1-\alpha; \\ \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{2\alpha^2} & if \quad -(1+\alpha)+\frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha; \\ \frac{1}{q-2\alpha}\left(p^B - p^A - \alpha\right) & if \quad 1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q}; \\ \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\,]}}{2\alpha^2} & if \quad -q-(1-\alpha) \leq p^A - p^B < 1+\alpha-q-\frac{2\alpha}{q}; \\ 1 & if \quad p^A - p^B < -q-(1-\alpha), \end{cases} \tag{4}$$

and $D_j^{B|VD}\left(p^A, p^B\right) = 1 - D_j^{A|VD}\left(p^A, p^B\right)$. Analogously, we denote by $D_j^{i|HD}\left(p^A, p^B\right)$, the demand of platform i in side j, under horizontal dominance, with:

$D_j^{A|HD}\left(p^A, p^B\right) =$

$$\begin{cases} 0 & if \quad p^A - p^B > 1-\alpha \\ \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{2\alpha^2} & if \quad 1-\alpha-q+\frac{q\alpha}{2} < p^A - p^B \leq 1-\alpha \\ \frac{1}{2}\left(1+\frac{p^B-p^A}{1-\alpha}-\frac{q}{2(1-\alpha)}\right) & if \quad -(1-\alpha)-\frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha-q+\frac{q\alpha}{2} \\ \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\,]}}{2\alpha^2} & if \quad -q-(1-\alpha) \leq p^A - p^B < -(1-\alpha)-\frac{q\alpha}{2} \\ 1 & if \quad p^A - p^B < -q-(1-\alpha) \end{cases}. \tag{5}$$

and $D_j^{B|HD}\left(p^A, p^B\right) = 1 - D_j^{A|HD}\left(p^A, p^B\right)$.

The demand functions $D_j^{A|VD}\left(p^A, p^B\right)$ and $D_j^{A|HD}\left(p^A, p^B\right)$ must be decreasing and continuous functions of the platforms' own prices. However, they are not globally concave [9].

In the particular case of vertical dominance, the demand is linear for price vectors $\left(p^A, p^B\right)$ such that:

$$1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q}.$$

When $-(1+\alpha)+\frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha$, the demand $D_j^{A|VD}\left(p^A, p^B\right)$ is strictly convex in

[9] The formal proof of these properties is straightforward, given the analytical expressions of demands (4) and (5).

p^A. Analogously, when:

$$-q-(1-\alpha) \leq p^A - p^B < 1+\alpha - q - \frac{2\alpha}{q},$$

the demand $D_j^{B|VD}\left(p^A, p^B\right)$ is strictly convex in p^B.

In the particular case of horizontal dominance, the demand is linear for price vectors $\left(p^A, p^B\right)$ such that:

$$-(1-\alpha) - \frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha-q+\frac{q\alpha}{2}.$$

When $1-\alpha-q+\frac{q\alpha}{2} \leq p^A - p^B < 1-\alpha$, the demand $D_j^{A|HD}\left(p^A, p^B\right)$ is strictly convex in p^A. Analogously, when:

$$-q-(1-\alpha) \leq p^A - p^B < -(1-\alpha) - \frac{q\alpha}{2},$$

the demand $D_j^{B|HD}\left(p^A, p^B\right)$ is strictly convex in p^B.

Given the demands faced by each platform in each sided of the market, it is now possible to obtain the profit of the two platforms as follows:

$$\pi^i(p^A, p^B) = p^i D_1^i(p^A, p^B) + p^i D_2^i(p^A, p^B),$$

where $D_j^i(p^A, p^B) = D_j^{A|VD}\left(p^A, p^B\right)$, in the case of vertical dominance, and $D^i(p^A, p^B) = D_j^{A|HD}\left(p^A, p^B\right)$, in the case of horizontal dominance.

The equations (4) and (5) imply symmetric demands in the two sides of the market:

$$D_1^i(p^A, p^B) = D_2^i(p^A, p^B) = D^i(p^A, p^B),$$

thus, the platforms' profit can be computed as follows:

$$\pi^i(p^A, p^B) = 2p^i D^i(p^A, p^B),\ i \in \{A, B\}.$$

Since the demand functions $D^i(p^A, p^B)$ are not globally concave, it is necessary to study under which conditions the profit of each platform is quasi-concave with respect to its access price.

Caplin and Nalebuff (1991) in general and, in particular, Neven and Thisse (1990) for the case of product differentiation in two dimensions, provide sufficient conditions for the existence of multiple equilibria in pure strategies for discrete choice models of differentiated product markets.

Without network externalities, our manuscript reduces to a model which satisfies the assumptions in Neven and Thisse (1990).

Since our model incorporates network effects, that are present for any given level of quality and variety, we must apply mathematical techniques to prove the quasi-concavity of the profit functions.[10] Nonetheless to say, the following proof is the central point of our manuscript.

Lemma 1 ***(Quasi-concavity of the profit function)***

For any pair combination (α, q), t*he profit functions of platforms* A *and* B *are quasi-concave in* p^A *and* p^B *both under vertical and horizontal dominance*, *respectively*.

Proof. *See Appendix B in section* 7.2. ■

This Lemma brings a relevant contribution to the current literature because proves that the payoff functions of the active players on the market are globally quasi-concave in a multidimensional product differentiation environment with the presence of network effects.

Therefore, the first order conditions of the platforms' maximization problem intersect at least once which guarantees that the first order conditions associated with the profit maximization problem are sufficient to characterize firms' optimal pricing choices[11].

We also examined the shape of the profit functions for a grid of values of the relevant parameters.

[10] To the best of our knowledge, our manuscript is the first proving the quasi-concavity of the profit function of the active platforms under the presence of network effects.

[11] Indeed, as you can check further, applying the theorem of Debreu, Glicksberg and Fan (1952) allows us to prove the existence of multiple interior pure-strategy Nash equilibria.

Figures 3(a) and 3(b) show a possible shape of the demand and figures 4(a) and 4(b) show a possible shape of the profit function of the low-quality platform A, for different levels of the inter-group externality under horizontal and vertical dominance, respectively.

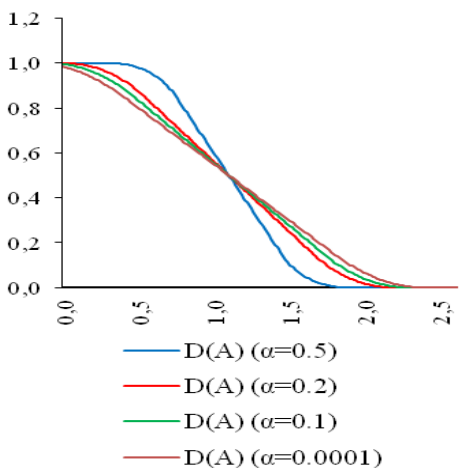

Figure 3(a)

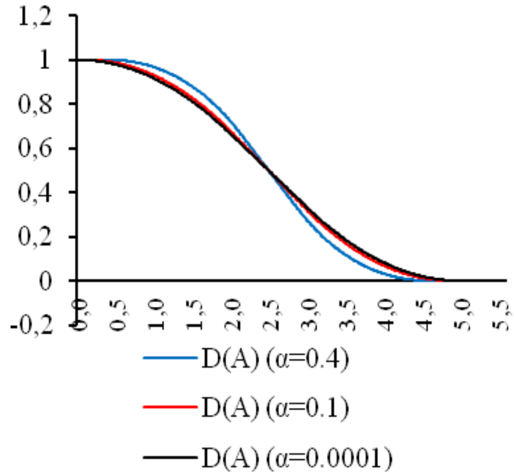

Figure 3(b)

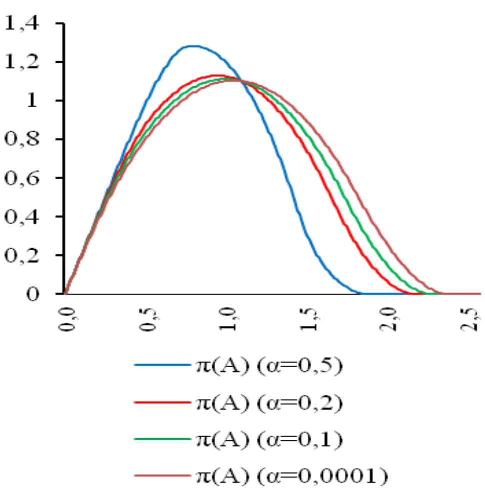

Figure 4(a)

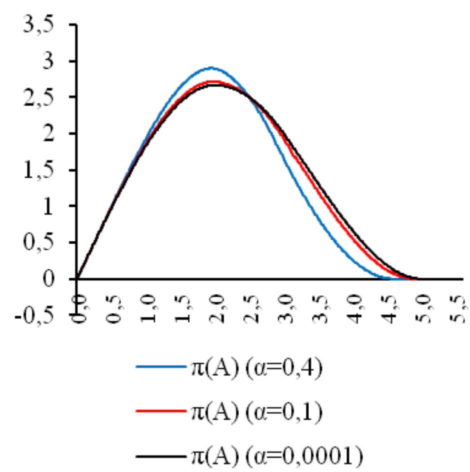

Figure 4(b)

As expected, the above figures reinforce the intuition that the profit functions are quasi-concavity on its prices domain.

4 Equilibria

In the first stage, platforms take decisions on access prices anticipating consumers behavior in the subsequent stage. Accordingly, the problem of platform i writes as follows:

$$\max_{p_i} 2p^i D^i(p^A, p^B),$$

with $i = A, B$ and $D^i(p^A, p^B) = D^{A|VD}\left(p^A, p^B\right)$ in the case of vertical dominance, and $D^i(p^A, p^B) = D^{A|HD}\left(p^A, p^B\right)$ in the case of horizontal dominance. Then, in line with Neven and Thisse (1990) [22], we should determine the equilibrium prices as a function of the product characteristics.

This leads us to distinguish between six types of equilibria, namely three under vertical dominance and three under horizontal dominance.

By other words, for each kind of dominance, we have the following types of equilibria:

(i) *case/subgame 1* - the equilibrium occurs on the linear segments of D^A and D^B;

(ii) *case/subgame 2* - the equilibrium occurs in the strictly convex segment of D^A and in the strictly concave segment of D^B;

(iii) *case/subgame 3* - the equilibrium occurs in the strictly concave segment of D^A and in the strictly convex segment of D^B.

We start by studying the price stage equilibrium for the linear segment of the demand of platforms A and B, for any combination of the parameters α and q (corresponding to case 1 when both firms are active in the market, which is always the case under Assumption 1).

4.1 Pure horizontal dominance equilibrium

We firstly describe the region where the equilibrium candidate under vertical dominance exists, corresponding to an equilibrium occurring under the circumstances described at figure 2(a).

Below, we fully characterize the correspondent equilibrium candidate.

Proposition 2 ***(Equilibrium under horizontal dominance)***

Let $q < \frac{6(1-\alpha)}{4-3\alpha}$. *The equilibrium outcomes are described as follows:*

$$\begin{cases} p^{A^*} = 1 - \alpha - \frac{q}{6},\ p^{B^*} = 1 - \alpha + \frac{q}{6}; \\ D^{A^*} = \frac{1}{2}\left[1 - \frac{q}{6(1-\alpha)}\right],\ D^{B^*} = \frac{1}{2}\left[1 + \frac{q}{6(1-\alpha)}\right]; \\ \pi^{A^*} = \frac{1}{36}\frac{[q-6(1-\alpha)]^2}{1-\alpha},\ \pi^{B^*} = \frac{1}{36}\frac{[q+6(1-\alpha)]^2}{1-\alpha}. \end{cases}$$

Proof. *See Appendix B in section* 7.2. ■

Comparing p^{A^*} and p^{B^*}, it can be easily checked that for any (α, q), we always obtain $p^{B^*} > p^{A^*}$, meaning that the access fee quoted by the high-quality platform is always higher than the access fee charged by the low-quality platform.

In horizontal dominance, equilibrium prices are affected by the intensity of inter-group network effects. It is also worth noting that the high-quality platform always has a higher market share than the low-quality platform, for any (α, q). Since the former also charges a higher price than the latter, we always have $\pi^{B^*} > \pi^{A^*}$.

4.2 Pure vertical dominance equilibrium

We firstly describe the region where the equilibrium candidate under vertical dominance exists, corresponding to an equilibrium occurring under the circumstances described at figure 2(b). Below, we fully characterize the correspondent equilibrium candidate.

Proposition 3 ***(Equilibrium under vertical dominance)***

Let $q > \frac{1}{2}\left(3 - 2\alpha + \sqrt{9 - 4\alpha(9-\alpha)}\right)$. *The equilibrium outcomes are described as follows:*

$$\begin{cases} p^{A^*} = \frac{q-\alpha}{3},\ p^{B^*} = \frac{2(q-\alpha)}{3}; \\ D^{A^*} = \frac{1}{3} - \frac{2\alpha}{3(q-2\alpha)},\ D^{B^*} = \frac{2}{3} + \frac{2\alpha}{3(q-2\alpha)}; \\ \pi^{A^*} = \frac{2(q-\alpha)(q-4\alpha)}{9(q-2\alpha)},\ \pi^{B^*} = \frac{8(q-\alpha)^2}{9(q-2\alpha)}. \end{cases}$$

Proof. *See Appendix B in section* 7.2. ■

Comparing p^{A^*} and p^{B^*}, it can be easily checked that for any (α, q), we always obtain $p^{B^*} > p^{A^*}$, meaning that the access fee quoted by the high-quality platform is always higher than the access fee charged by the low-quality platform. In vertical dominance, equilibrium prices are affected by the intensity of inter-group network effects. However, the downward pressure over prices is stronger under horizontal dominance. The intuition behind this result is analyzed in the section 5. It is also worth noting that the high-quality platform always has a higher market share than the low-quality platform, for any (α, q). Since the former also charges a higher price than the latter, we always have $\pi^{B^*} > \pi^{A^*}$.

4.3 Intermediate equilibrium

After characterizing the equilibrium candidates emerging from the interaction of the linear segments of the demands of platforms A and B (*case 1*), we verify the other interior equilibrium candidates emerging from case 2, for each type of dominance, corresponding to the circumstances described in figure $1(b)$.[12]. Figure 5 shows the price domain conditions under pure vertical dominance (blue region) and under pure horizontal dominance (yellow region), where each equilibrium candidate is an effective equilibrium. The white region corresponds to the equilibrium candidate occurring under the circumstances described in

[12] As Neven and Thisse (1990) point out, since platform A is the low-quality platform, a situation described as in figure $17(d)$ never arises at the equilibrium. This implies, in our manuscript, that under horizontal dominance for $-(1-\alpha) - q \leq p^A - p^B < (1-\alpha) - q$ and under vertical dominance for $-q - (1-\alpha) \leq p^A - p^B < 1 + \alpha - q - \frac{2\alpha}{q}$, there is no price equilibrium since the prices solving the first order conditions are incompatible with the corresponding price domain condition.

figure 1(b).

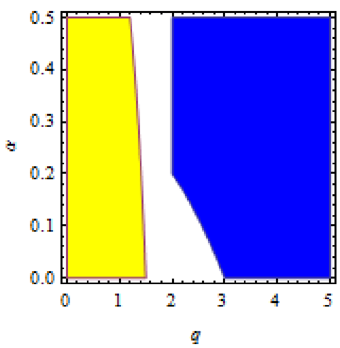

Figure 5

For such values, the equilibrium occurs in the strictly convex segment of D^A and in the strictly concave segment of D^B. The profit of platform A and platform B are given by:

$$\pi^A = \frac{p^A\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q\left[q-2\alpha(1-\alpha+p^B-p^A)\right]}\right]}{\alpha^2},$$

$$\pi^B = 2p^B\left\{1-\frac{\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q\left[q-2\alpha(1-\alpha+p^B-p^A)\right]}\right]}{2\alpha^2}\right\}.$$

Given the quasi-concavity of the profit function, an equilibrium price candidate is always an equilibrium as long as the price domain condition is verified. Under horizontal dominance, the equilibrium is verified for:

$$1-\alpha-q+\tfrac{q\alpha}{2} < p^{A*}-p^{B*} \leq 1-\alpha,$$

and under vertical dominance, the equilibrium is verified for:

$$-(1+\alpha)+\tfrac{2\alpha}{q} < p^{A*}-p^{B*} \leq 1-\alpha.$$

We are not able to analytically solve the model for this "intermediate" equilibrium candidate[13].

However, the proposed theorem of Debreu, Glicksberg and Fan (1952) allows us to conclude that an equilibrium exists for this case (see, for example, pp. 34, of Fudenberg and Tirole, 1991).

[13] See Appendix C of section 7.3 for details on the best reply functions of both platforms.

Theorem 4 ***(Debreu, Glicksberg and Fan (1952))*** *Consider a strategic-form game: (i) whose strategy spaces S_i are nonempty compact convex subsets of an Eucledian space. If the payoff functions are (ii) continuous and (iii) quasi-concave in s_i, there exists a pure-strategy Nash equilibrium.*[14]

The proof towards Lemma *1* ratifies the above Theorem in a context of a two-dimensional product differentiation environment with the presence of network effects on the market. The following conclusion is, therefore, straightforward.

Remark 1 ***(Intermediate equilibrium)***

For any pair combination (α, q), there exists a price equilibrium candidate in the strictly convex segment of D^A and in the strictly concave segment of D^B. The equilibrium candidate must be an equilibrium:

(i) *under horizontal dominance, for* $\frac{6(1-\alpha)}{4-3\alpha} < q < 2$;

(ii) *under vertical dominance, for* $2 < q < \frac{1}{2}\left(3 - 2\alpha + \sqrt{9 - 4\alpha(9-\alpha)}\right)$.

Proof. *First, the existence of a price equilibrium candidate for such domain region is proved in Theorem 4 (and observe the proof of Lemma 1 relatively to the convex (concave) segment of the profit function of platform A (platform B) on each type of dominance); secondly, the boundaries on each type of dominance result from Proposition* 2 *and Proposition* 3. ■

5 Comparative Statics

In this section we analyze how the magnitude of the quality gap q, and the intensity of the inter-group externality α, affect equilibrium outcomes.

[14] Debreu (1952) used a generalization of this theorem to prove that competitive equilibria exist when consumers have quasi-convex preferences.

5.1 Impact of the quality gap

Regarding the impact of the quality gap on equilibrium prices, we obtain that the price of the high-quality platform always increases with the magnitude of the quality gap. Everything else the same, an increase in q makes platform B relatively more attractive than platform A and, therefore, firm B is able to increase its price for all (q, α).

As far as concerns firm A, under horizontal dominance, an increase in q reduces the price of firm A. Since q is not too high (meaning that products are not very differentiated on the vertical dimension), the low-quality firm reduces its price after an increase in q.

In contrast, under vertical dominance, the price of the low-quality gap also increases with q. In that case, products are already significantly differentiated in the vertical dimension and, thus, an increase in q makes products even more differentiated, relaxing price competition, which is line with standard vertical differentiation literature (see, for example, Shaked and Sutton (1982). Despite the fact that both prices are increasing with q, the price of the high-quality platform always increases more than the price of its rival since $\frac{\partial p^{B^*}}{\partial q} > \frac{\partial p^{A^*}}{\partial q} > 0$.

Regarding the impact of the quality gap on equilibrium market shares, we obtain that under horizontal dominance, we have $\frac{\partial D^{A^*}}{\partial q} = -\frac{1}{12(1-\alpha)} < 0$, implying $\frac{\partial D^{B^*}}{\partial q} > 0$ (since the market is full covered). Following an increase in q, the high-quality platform becomes more attractive than the low-quality platform and it is able to increase its market share, despite the fact that the price gap $p^{B^*} - p^{A^*}$ is increasing with α. However, when the quality gap is very large, it follows $\frac{\partial D^{A^*}}{\partial q} = \frac{2\alpha}{3(q-2\alpha)^2} > 0$, implying $\frac{\partial D^{B^*}}{\partial q} < 0$. This is so, because the high-quality platform increases its price substantially more than the low-quality platform.

Lemma 4 pp.8, from Shaked and Sutton (1982) establishes that "*revenues of both firms increase as the quality of the better product improves*".

Considering the horizontal dominance environment and evaluating the impact of q on the equilibrium profits of platforms A and B, we get that: (i) for $q \in \left(0, \frac{3}{2}(1-\alpha)\right)$ follows:

$$\frac{\partial \pi^{A^*}}{\partial q} < 0; \frac{\partial \pi^{B^*}}{\partial q} > 0,$$

and (ii) for $q \in \left(\frac{3}{2}(1-\alpha), \frac{6(1-\alpha)}{4-3\alpha}\right)$ we obtain that:

$$\frac{\partial \pi^{A^*}}{\partial q} > 0; \frac{\partial \pi^{B^*}}{\partial q} > 0.$$

Then, under horizontal dominance ($q < 2$) and when the quality gap is low, the revenue of the high-quality platform B increases as the quality of the better product improves and the revenue of the low-quality platform A decreases as the quality of the better product improves. However, when the quality gap is sufficiently strong, the revenue of the high-quality platform B increases as the quality of the better product improves and the revenue of the low-quality platform A also increases as the quality of the better product improves.

This result is not only distinct from Shaked and Sutton (1982) but also complements Neven and Thisse (1990) since they did not take into consideration the presence of network effects influencing simultaneously both variety and quality.

This evidence suggests that when the high-quality platform B puts an excessive effort on increasing the vertical differentiation between platforms in a horizontal dominance environment constitutes a strategy that benefits it's rival, the low-quality platform A.

Under vertical dominance, our results are similar to the findings of Shaked and Sutton (1982). The equilibrium profit of the low-quality platform A is always increasing with q (since the equilibrium price and market share are increasing with q).However, at platform B, the market shares are decreasing in q. However, it follows that the positive price effect more than compensates the negative demand effect of the high-quality platform.

Lemma 5 ***(Impact of vertical differentiation)***

Under horizontal dominance:

(i) *for* $q < \frac{3}{2}(1-\alpha)$ *follows that* $\frac{\partial \pi^{A^*}}{\partial q} < 0$;

(ii) *for* $q \in \left(\frac{3}{2}(1-\alpha), \frac{6(1-\alpha)}{4-3\alpha}\right)$ *follows that* $\frac{\partial \pi^{A^*}}{\partial q} > 0$.

Proof. *See Appendix B in section* 7.2. ■

Intuitively, for sufficiently lower levels of the quality gap between platforms, when the quality of the better product marginally increases leads to a reduction on the profit of the low-quality platform because on that case we are in an "environment" of horizontal dominance such that agents on both sides of the market attribute much more value to variety rather than quality.

Thus, on this case, the result of Shaked and Sutton (1982) is disrupt. Neven and Thisse (1990) are not able to capture our finding since their model does not consider the presence of inter-group externalities.

5.2 Impact of the intensity of the inter-group externality

In what concerns the impact of the intensity of the inter-group network effect on equilibrium prices, we obtain that the standard results in two-sided markets hold. In particular, under horizontal dominance, we have that:

$$\frac{\partial p^{A^*}}{\partial \alpha} = \frac{\partial p^{B^*}}{\partial \alpha} = -1.$$

In line with Armstrong (2006), when the inter-group network effect is sufficiently weak consumers benefit from lower prices due to inter-group network effects (when these effects take place, firms set discounts in one side of the market with the objective of enhancing demand in opposite side of the market). In line with the seminal literature on two sided markets, we obtain that such price reduction is increasing with the intensity of network effects.

Under vertical dominance, the equilibrium prices are also negatively affected by the intensity of the inter-group network effect. However, prices decrease less relatively to the case of horizontal dominance and the price of high-quality platform is more responsive relatively to a change on the level of the inter-group externality since:

$$\frac{\partial p^{A^*}}{\partial \alpha} = -\frac{1}{3}; \ \frac{\partial p^{B^*}}{\partial \alpha} = -\frac{2}{3}.$$

Regarding the impact of the inter-group network intensity on equilibrium market shares, we obtain that for horizontal dominance yields $\frac{\partial D^{A^*}}{\partial \alpha} < 0$, implying $\frac{\partial D^{B^*}}{\partial \alpha} > 0$, since $D^{B^*} = 1 - D^{A^*}$. More precisely, $\frac{\partial D^{A^*}}{\partial \alpha} = -\frac{1}{12}\frac{q}{(\alpha-1)^2} < 0$.

Hence, an increase in the intensity of inter-group network externalities leads increases market concentration, favoring the high-quality platform. When α increases, consumers value more the number of consumers (on the other side) participating in each platform and therefore the high-quality platform becomes more attractive than the rival (recall that the former always has a higher market share than the low quality platform).

For vertical dominance, equilibrium market shares are responsive to the inter-group externality on the same way since $\frac{\partial D^{A^*}}{\partial \alpha} < 0$ and $\frac{\partial D^{B^*}}{\partial \alpha} > 0$.

Regarding the impact of α on equilibrium profit, for $q \leq \frac{6(1-\alpha)}{4-3\alpha}$, it is easy to see that the equilibrium profit of the low-quality platform is decreasing with α (since both the price and the market share of the platform decrease with the intensity of the inter-group network effect).

In the case of platform B, the two effects are moving in opposite direction but the price effect is dominant and the profit of platform B is also decreasing with α. Thus, it follows that $\frac{\partial \pi^{A^*}}{\partial \alpha} < 0$ and $\frac{\partial \pi^{B^*}}{\partial \alpha} < 0$.

For vertical dominance, the equilibrium profit of the low-quality platform is decreasing with α (since both the price and the market share of the platform decrease with the intensity of the inter-group network effect).

In the case of platform B, the two effects are moving in opposite direction. The dominant effect here is the demand effect and, thus, the profit of platform B is increasing with α.

Accordingly, the following Lemma holds.

Lemma 6 ***(Impact of the inter-group externality)***

Under vertical dominance, the profit of the high-quality platform B is increasing with an increment on the intensity of the network effect.

Proof. *See Appendix B in section* 7.2. ■

The fact that equilibrium profits are decreasing with α is in line with Armstrong (2006) and it results from the fact that an increase in the intensity of the inter-group network effect intensifies price competition between the two platforms. However our model disrupts this standard result of two-sided markets.

Intuitively, for sufficiently higher levels of the quality gap between platforms, we fall in the case of pure vertical dominance, that is, agents on both sides of the market attribute much more value to quality rather than variety. Thus, it is not surprising that the active platform with higher quality is the only platform that, under these circumstances, benefits with a marginally increment on the intensity of the network effect.

Obviously, Armstrong (2006) does not capture this effect since the adaptation of Hotelling (1929) model does not able researchers to analytically capture the key distinction between vertical and horizontal dominance, which can be achievable using Neven and Thisse (1990) model.

6 Conclusions

This paper analyzes price competition between verti-zontally differentiated platforms. In light of this, we extend the pure horizontal model of two-sided markets presented by Armstrong (2006), allowing for vertical differentiation between the platforms.

Our verti-zontal differentiation set-up is built on a simplified version of the model proposed by Neven and Thisse (1990). While the latter, allows for quality choice and endogenous location, we take these choices as exogenously given, allowing instead for the existence of inter-group network effects. Our equilibrium analysis shows that equilibrium outcomes depend on the intensity of the inter-group network effects vis-à-vis the gap in the quality of the platforms.

Considering the case in which price discrimination between sides is not allowed, we find that regardless of the intensity of network effects, we find that the high-quality platform always quotes a higher price and attracts a larger fraction of consumers than the low-quality platform. The intuition of this result is that, with a two-dimensional product differentiation and considering only a discriminatory regime on the quality attribute, a low-quality platform cannot be a dominant intermediary on the market.

Moreover we find that, under horizontal dominance, the profit of the low-quality platform decreases as the quality of the better product improves when the quality gap between platforms is sufficiently small. This result contrasts with Shaked and Sutton (1982) seminal contribution.

Finally, under vertical dominance, we find that the profit of the high-quality platform is increasing with an increment on the intensity of the network effect, which disrupts the evidence from Armstrong (2006) in which network effects tend to put a downward pressure over equilibrium profits of both active rivals. This is due to the fact that we incorporate an exogenous discrimination on quality between the two platforms.

Other extensions of this model are worthwhile as well, addressing issues such as (partial) multi-homing, cost asymmetries, the role of imperfect information, dynamic pricing, endogenous locations and qualities, among others.

7 Appendix

7.1 Appendix A - Demand functions

Consider the case of vertical dominance. The configuration of demand for different price vectors $\left(p^A, p^B\right)$ is illustrated in figure 1 (of section 3). The figure shows that the specification of the demand function, depends on the magnitude of the price gap $p^A - p^B$.

First, notice that from expressions (2) and (3), follows the extreme situations:

$$\begin{cases} \widetilde{x}_j\left(0\right) = \frac{1}{2} + \frac{p^B - p^A}{2} + \frac{\alpha(2D_k^A - 1)}{2}; \\ \widetilde{x}_j\left(1\right) = \frac{1}{2} + \frac{p^B - p^A}{2} + \frac{\alpha(2D_k^A - 1)}{2} - \frac{q}{2}; \\ \widetilde{y}_j\left(0\right) = \frac{1}{q} + \frac{p^B - p^A}{q} + \frac{\alpha(2D_k^A - 1)}{q}; \\ \widetilde{y}_j\left(1\right) = \frac{1}{q} + \frac{p^B - p^A}{q} + \frac{\alpha(2D_k^A - 1)}{q} - \frac{2}{q}. \end{cases}$$

In figure $1(a)$, all consumers prefer to buy from platform B. In that case, the price gap must high enough so that $\widetilde{y}_j\left(0\right) < 0$, or equivalently, $p^A - p^B > 1 - \alpha$. For price vectors $\left(p^A, p^B\right)$ such that $p^A - p^B > 1 - \alpha$, we have $D_j^{A|VD}\left(p^A, p^B\right) = 0$.

In figure $1(b)$ some consumers start buying from platform A, although the ones located closer to platform B, always prefer this firm, regardless of their willingness to pay for quality, $y \in [0, 1]$. In that case, the price gap $p^A - p^B$ must be such that $0 \leq \widetilde{y}_j\left(0\right) < 1$ and $\widetilde{y}_j\left(1\right) < 0$. These inequalities impose the follow condition on the price gap in the case of vertical dominance:

$$-(1 - \alpha) + \frac{2\alpha}{q} < p^A - p^B \leq 1 - \alpha, \tag{6}$$

with $q > 2$ under vertical dominance. When (6) holds, the demand specification is given by:

$$\begin{aligned} D_j^{A|VD}\left(p^A, p^B, D^A\right) &= \frac{\widetilde{x}_j(0) \times \widetilde{y}_j(0)}{2} \Leftrightarrow \\ D_j^{A|VD}\left(p^A, p^B, D^A\right) &= \frac{1}{q}\left(\frac{1}{2} + \frac{p^B - p^A}{2} + \alpha D_j^{A|VD} - \frac{\alpha}{2}\right)^2 \Leftrightarrow \\ D_j^{A|VD}\left(p^A, p^B, D^A\right) &= \frac{1}{4q}\left[1 + \alpha\left(2D_j^{A|VD} - 1\right) + p^B - p^A\right]^2. \end{aligned}$$

Solving with respect to $D_j^{A|VD}$ yields:

$$D_j^{A|VD}\left(p^A, p^B\right) = \frac{q - \alpha\left(p^B - p^A + (1 - \alpha)\ \right) \pm \sqrt{q[q - 2\alpha(p^B - p^A + 1 - \alpha\ )]}}{2\alpha^2}.$$

The simulation demonstrates that, for the convex segment of the demand of the low-quality platform, the specification of the demand is given by:

$$D_j^{A|VD}\left(p^A,p^B\right)=\frac{q-\alpha\left(p^B-p^A+(1-\alpha)\ \right)-\sqrt{q[q-2\alpha(p^B-p^A+1-\alpha\ )]}}{2\alpha^2}.$$

In figure 1(c), we observe that consumers with higher willingness to pay for quality prefer platform B over platform A. However, there are consumers in all locations $x\in[0,1]$ participating in platform A. Accordingly, the following conditions must be verified: $\widetilde{y}_j\left(0\right)\leq 1$ and $\widetilde{y}_j\left(1\right)\geq 0$, leading to the following conditions on the price gap:

$$1+\alpha-q-\tfrac{2\alpha}{q}\leq p^A-p^B\leq -(1-\alpha)+\tfrac{2\alpha}{q}.$$

In that case, the demand specification is given by:

$$D_j^{A|VD}\left(p^A,p^B\right)=\tfrac{\widetilde{y}_j(0)+\widetilde{y}_j(1)}{2}=\tfrac{1}{q-2\alpha}\left(p^B-p^A-\alpha\right).$$

In figure 1(d), the price gap p^A-p^B is such that all consumers located closer to firm A, prefer to participate in this platform, regardless of their willingness to pay for quality $y\in[0,1]$. Hence, $\widetilde{y}_j\left(0\right)>1$ and $0<\widetilde{y}_j\left(1\right)\leq 1$, yielding:

$$-q-(1-\alpha)\leq p^A-p^B<1+\alpha-q-\tfrac{2\alpha}{q}.$$

In that case, the demand specification can be obtained by computing:

$$\begin{aligned}D_j^{A|VD}\left(p^A,p^B,D^A\right) &= 1-\tfrac{[1-\widetilde{x}_j(1)]\times[1-\widetilde{y}_j(1)]}{2}\Leftrightarrow\\ D_j^{A|VD}\left(p^A,p^B,D^A\right) &= 1-\tfrac{1}{4q}\left\{\left[1+p^A-p^B+\alpha\left(1-D_j^{A|VD}\right)\right]^2-q^2\right\}.\end{aligned}$$

Solving with respect to $D_j^{A|VD}$ yields:

$$D_j^{A|VD}\left(p^A,p^B\right)=\frac{\alpha\left(p^A-p^B+1+\alpha+q\ \right)-q\pm\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q\ )]}}{2\alpha^2}.$$

The simulation demonstrates that, for the convex segment of the demand of the low-quality platform, the specification of the demand is given by:

$$D_j^{A|VD}\left(p^A,p^B\right)=\frac{\alpha\left(p^A-p^B+1+\alpha+q\ \right)-q+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q\ )]}}{2\alpha^2}.$$

Finally, in figure 1(e) all consumers prefer to participate in platform A, yielding:

$$D_j^{A|VD}\left(p^A,p^B\right)=1.$$

For this to occur, we must have $\widetilde{y}_j(1) > 1$, yielding $p^A - p^B < -q-(1-\alpha)$. In light of this, we have that the demand function $D_j^{A|VD}\left(p^A,p^B\right)$ under vertical dominance, corresponds to a piecewise function with the following specification:

$$D_j^{A|VD}\left(p^A,p^B\right) =$$

$$\begin{cases} 0 & if \quad p^A-p^B > 1-\alpha; \\ \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{2\alpha^2} & if \quad -(1+\alpha)+\frac{2\alpha}{q} < p^A-p^B \leq 1-\alpha; \\ \frac{1}{q-2\alpha}\left(p^B-p^A-\alpha\right) & if \quad 1+\alpha-q-\frac{2\alpha}{q} \leq p^A-p^B \leq -(1+\alpha)+\frac{2\alpha}{q}; \\ \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\,]}}{2\alpha^2} & if \quad -q-(1-\alpha) \leq p^A-p^B < 1+\alpha-q-\frac{2\alpha}{q}; \\ 1 & if \quad p^A-p^B < -q-(1-\alpha). \end{cases}$$

Since each side of the market is assumed to be covered by the two firms, platform's B demand function is simply:

$$D_j^{B|VD}\left(p^A,p^B\right) = 1 - D_j^{A|VD}\left(p^A,p^B\right).$$

The same comments apply, *mutatis mutandis*, when obtaining the specification of the demand function under horizontal dominance. In particular, it is easy to see that the monopoly outcomes occur under exactly the same conditions. Similarly, when all consumers located closer to platform B (resp. platform A) participate in this platform, regardless of their willingness to pay for qualities (i.e. the configuration of firms' market shares under horizontal dominance is similar to the one depicted in figure 1(b) (resp. figure 1(d) under vertical dominance), the computation of platforms' demand under horizontal dominance is analogous to the case of vertical dominance. In particular, we obtain that the expressions of demands are exactly the same as in the case of vertical dominance, although, the corresponding price domain is different, since $q < 2$ under horizontal dominance. Accordingly, we have that:

$$D_j^{A|HD}\left(p^A,p^B\right) = \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q\left(q-2\alpha(1-\alpha+p^B-p^A)\,\right)}}{2\alpha^2},$$

when $1-\alpha-q+\frac{q\alpha}{2} \leq p^A-p^B < 1-\alpha$, and:

$$D_j^{A|HD}\left(p^A,p^B\right) = \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q\left[q-2\alpha(1-\alpha-p^B+p^A+q)\,\right]}}{2\alpha^2},$$

when $-q-(1-\alpha) \leq p^A - p^B < -(1-\alpha) - \frac{q\alpha}{2}$.

The situation described in figure 1(c) for the case of vertical dominance does not arise in the case of horizontal dominance. Instead, when $-(1-\alpha)-\frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha-q+\frac{q\alpha}{2}$, the configuration of demand is depicted in figure 2(a). In that case, all consumers located closer to platform A (resp. platform B) participate in this platform (regardless of their willingness to pay for quality) [15]. This is the case, when $\widetilde{y}_j(0) > 1$ and $\widetilde{y}_j(1) < 0$, yielding:

$$-(1-\alpha) - \frac{q\alpha}{2} \leq p^A - p^B \leq 1 - \alpha - q + \frac{q\alpha}{2}.$$

The expression of demand can be obtained as follows:

$$\begin{aligned} D_j^{A|HD}\left(p^A, p^B\right) &= \frac{\widetilde{x}_j(0)+\widetilde{x}_j(1)}{2} \Leftrightarrow \\ D_j^{A|HD}\left(p^A, p^B\right) &= \frac{1}{2}\left[1 + \frac{p^B - p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right]. \end{aligned}$$

Accordingly, the demand function $D_j^{A|HD}\left(p^A, p^B\right)$ under horizontal dominance, corresponds to a piecewise function with the following specification:

$$D_j^{A|HD}\left(p^A, p^B\right) =$$

$$\begin{cases} 0 & if \quad p^A - p^B > 1-\alpha; \\ \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{2\alpha^2} & if \quad 1-\alpha-q+\frac{q\alpha}{2} \leq p^A - p^B < 1-\alpha; \\ \frac{1}{2}\left[1+\frac{p^B-p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right] & if \quad -(1-\alpha)-\frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha-q+\frac{q\alpha}{2}; \\ \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}{2\alpha^2} & if \quad -q-(1-\alpha) \leq p^A - p^B < -(1-\alpha)-\frac{q\alpha}{2}; \\ 1 & if \quad p^A - p^B < -q-(1-\alpha), \end{cases} .$$

and $D_j^{B|HD}\left(p^A, p^B\right) = 1 - D_j^{A|HD}\left(p^A, p^B\right)$.

□

7.2 Appendix B - Proofs of Propositions and Lemmas

Proof of Lemma 1

[15] Recall that in the case of horizontal dominance, consumers attach more importance to horizontal differentiation vis-à-vis vertical differentiation.

Let us consider the profit of platform A :

$$\pi^A\left(p^A, p^B\right) = 2p^A D^A(p^A, p^B),$$

with $D^A(p^A, p^B) = D^{A|VD}\left(p^A, p^B\right)$ under vertical dominance, and $D^A(p^A, p^B) = D^{A|HD}\left(p^A, p^B\right)$ under horizontal dominance. Recall that the demand of platform i is a continuous and non-increasing function of p^i, both under horizontal and vertical dominance.

A. proof under Horizontal Dominance

In the case of horizontal dominance we obtain that the profit function of platform A is given by:

$\pi^{A|HD}\left(p^A, p^B\right) =$

$$\begin{cases} 0, p^A - p^B > 1-\alpha; \text{ (branch 5)} \\ \frac{p^A\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}\right]}{\alpha^2}, 1-\alpha-q+\frac{q\alpha}{2} < p^A - p^B \leq 1-\alpha; \text{ (branch 4)} \\ p^A\left[1+\frac{p^B-p^A}{1-\alpha}-\frac{q}{2(1-\alpha)}\right], -(1-\alpha)-\frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha-q+\frac{q\alpha}{2}; \text{ (branch 3)} \\ \frac{p^A\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\,]}\right]}{\alpha^2}, \; -q-(1-\alpha) \leq p^A - p^B < -(1-\alpha)-\frac{q\alpha}{2}; \text{ (branch 2)} \\ 2p^A, p^A - p^B < -q-(1-\alpha).\text{(branch 1)} \end{cases}$$

(i) **Concave and linear segments of the profit function** (*branches 1-3 and 5*)

Furthermore, in the case of horizontal dominance, $D^{A|HD}\left(p^A, p^B\right)$ is concave and decreasing for $p^A - p^B \leq 1-\alpha-q+\frac{q}{2}\alpha$ and it is constant for $p^A - p^B > 1-\alpha$. Since demand in those price domains is concave and non-increasing, the profit function of platform A is also a concave function of p^A in that price domain[16]. Note that for the concave segment

[16]Note that since $\pi^A\left(p^A, p^B\right) = 2p^A D^A(p^A, p^B)$, follows that $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 2D^A(p^A, p^B) + 2p^A \frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A}$. Thus, $\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2} = 4\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A} + 2p^A \frac{\partial^2 D^A\left(p^A,p^B\right)}{\partial (p^A)^2}$. Since the demand is (i) de-

$-q-(1-\alpha) \leq p^A - p^B < -(1-\alpha) - \frac{q\alpha}{2}$ of the demand of platform A yields:

$$\begin{aligned}
\frac{\partial D^{A|HD}\left(p^A,p^B\right)}{\partial p^A} &= \frac{1}{2\alpha} - \frac{\sqrt{q\left[q-2\alpha(1-\alpha+p^A-p^B+q)\ \right]}}{2\alpha\left(q-2\alpha(1-\alpha+p^A-p^B+q)\ \right)}; \\
\frac{\partial^2 D^{A|HD}\left(p^A,p^B\right)}{\partial (p^A)^2} &= -\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}} < 0.
\end{aligned}$$

The profit function of platform A is linear at branch 1 and concave at branches 2 and 3. The first derivative of the profit function relatively to its price at branches 2 and 3 are, respectively, given by:

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 1 + \frac{1}{\alpha} - \frac{q}{\alpha^2} + \frac{2p^A}{\alpha} + \frac{q}{\alpha} - \frac{p^B}{\alpha} + \frac{\sqrt{q[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]}}{\alpha^2} - \frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]};$$

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = \frac{2(1-\alpha)+2p^B-4p^A-q}{2(1-\alpha)}.$$

Evaluating the derivatives at the kink $p^A = p^B - (1-\alpha) - \frac{q\alpha}{2}$, follows that the LHS derivative (using the profit function expression of branch 2) and the RHS derivative (using the profit function expression of branch 3) are given by:

$$\begin{aligned}
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B-(1-\alpha)-\frac{q\alpha}{2}\right]^-} &= 3-q+\frac{q}{2(1-\alpha)} - \frac{p^B}{1-\alpha} + \frac{1-\alpha+q\alpha}{\alpha^2}; \\
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B-(1-\alpha)-\frac{q\alpha}{2}\right]^+} &= 3-q+\frac{q}{2(1-\alpha)} - \frac{p^B}{1-\alpha}.
\end{aligned}$$

In the frontier between the two concave segments of the profit function, the following inequality must hold:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B-(1-\alpha)-\frac{q\alpha}{2}\right]^-} \geq \left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B-(1-\alpha)-\frac{q\alpha}{2}\right]^+}.$$

By words, the LHS derivative at the kink point $p^A = p^B - (1-\alpha) - \frac{q\alpha}{2}$ must not be lower than the RHS derivative to obtain a concave profit function in the domain $-q-(1-\alpha) \leq p^A - p^B \leq 1-\alpha-q+\frac{q\alpha}{2}$. We verify that this inequality is verified, since the LHS derivative is higher than the RHS derivative at $p^A = p^B - (1-\alpha) - \frac{q\alpha}{2}$ because:

$$1-\alpha+q\alpha \geq 0,$$

creasing and (ii) concave at the considered segment follows that (i) $\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A} < 0$ and (ii) $\frac{\partial^2 D^A\left(p^A,p^B\right)}{\partial (p^A)^2}$, respectively. Then, we obtain $\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2} < 0$.

holds under assumption 1.

(ii) **Convex segment of the profit function** (*branch 4*)

By contrast, the only domain in which the profit function is not concave is:

$$1-\alpha-q+\tfrac{q}{2}\alpha<p^A-p^B\leq 1-\alpha,$$

where the demand function $D^{A|HD}\left(p^A,p^B\right)$ is strictly convex, with:

$$\frac{\partial^2 D^{A|HD}\left(p^A,p^B\right)}{\partial\left(p^A\right)^2}=\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}}>0$$

for such values of $\left(p^A,p^B\right)$.

In that price domain, yields that the profit function is given by:

$$\pi^A\left(p^A,p^B\right)=2p^AD^A(p^A,p^B)=p^A\left[\frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha^2}\right],$$

The derivative relatively to price is given by:

$$\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}=-\frac{\left(\alpha p^B-q-2\alpha p^A-\alpha^2+\alpha\right)\sqrt{\rho}-2\alpha qp^B+3\alpha qp^A+q^2-2\alpha q(1-\alpha)}{\alpha^2\sqrt{\rho}}\quad(7)$$

with $\rho=-2\alpha q\left(p^B-p^A\ \right)+q^2-2\alpha(1-\alpha)q$. Alternatively, expression (7) can be re-written as:

$$\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}=1+\frac{q}{\alpha^2}-\frac{1+p^B-2p^A}{\alpha}-\frac{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha^2}-\frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]}.\quad(8)$$

The second derivative of π^A relatively to p^A in the mentioned price domain is given by:

$$\frac{\partial^2\pi^A\left(p^A,p^B\right)}{\partial\left(p^A\right)^2}=\frac{2}{\alpha}+\frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]^2}-\frac{2\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]},$$

and the third derivative of the profit function with respect to their price is given by:

$$\frac{\partial^3\pi^A\left(p^A,p^B\right)}{\partial\left(p^A\right)^3}=\frac{3\sqrt{q\left[q-2\alpha(1-\alpha+p^B-p^A)\ \right]}\left\{q-\alpha\left[2(1-\alpha)+2p^B-p^A\right]\ \right\}}{\left[q-2\alpha(1-\alpha+p^B-p^A)\ \right]^3}.$$

Thus, it is straightforward that $\frac{\partial^3\pi^A\left(p^A,p^B\right)}{\partial\left(p^A\right)^3}>0$ under assumption 1, so that $\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}$ is strictly convex in p^A for the price domain $1-\alpha-q+\frac{q}{2}\alpha<p^A-p^B\leq 1-\alpha$.

Also considering the behavior of $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}$ in the neighborhood of $p^A = p^B + (1-\alpha)$, which corresponds to the upper bound of the convex segment of the demand, plugging $p^A = p^B + (1-\alpha)$ in equation (7), we obtain:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=[p^B+(1-\alpha)]^-} = -\frac{\left[q+\alpha(1-\alpha+p^B)\ \right]\left(\sqrt{q^2}-q\right)}{q\alpha^2} = 0.$$

Since the LHS derivative of π^A relatively to p^A is null at $p^A = \left[p^B + (1-\alpha)\right]^-$ and the second derivative of the profit function evaluated at $p^A = \left[p^B + (1-\alpha)\right]^-$ is given by:

$$\left.\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2}\right|_{p^A=[p^B+(1-\alpha)]^-} = \frac{p^B+1-\alpha}{q} > 0,$$

it follows that the profit function $\pi^A\left(p^A,p^B\right)$ reaches to a minimum at the point $p^A = p^B+(1-\alpha)$ and, thus, $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 0$ has at most one solution in the domain $1-\alpha-q+\frac{q\alpha}{2} < p^A - p^B \leq 1-\alpha$.

(iii) **Sign of the derivatives at the kink between the linear and convex segment of the demand function**

Furthermore, in the neighborhood of $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$, the lower bound of the convex segment of the demand function, the RHS and the LHS derivatives at the kink point $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$ of the profit function of platform A are equal.

(a) To compute the RHS derivative, we plug $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$ in equation (7) to obtain:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B+1-\alpha-q+\frac{q}{2}\alpha\right]^+} = -1 - \frac{p^B}{1-\alpha} + \frac{q}{2}\left(\frac{3-2\alpha}{1-\alpha}\right).$$

(b) To compute the LHS derivative note that, for $\left[p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha\right]^-$, the profit of platform A is equal to:

$$\pi^A\left(p^A,p^B\right) = p^A\left[1 + \frac{p^B-p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right].$$

Then, the derivative is given by:

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = \frac{2(1-\alpha)+2p^B-4p^A-q}{2(1-\alpha)}.$$

Evaluating the derivative at $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$, we obtain the LHS derivative in the neighborhood of $p^A = \left[p^B + 1 - \alpha - q + \frac{q}{2}\alpha\right]^-$:

$$\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial p^A}\rceil_{p^A=\left[p^B+1-\alpha-q+\frac{q}{2}\alpha\right]^-} = -1 - \frac{p^B}{1-\alpha} + \frac{q}{2}\left(\frac{3-2\alpha}{1-\alpha}\right).$$

Thus, the sign of $\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial p^A}$ is the same on both sides of $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$ and both derivatives are equal:

$$\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\rceil_{p^A=\left[p^B+1-\alpha-q+\frac{q}{2}\alpha\right]^+} = \frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\rceil_{p^A=\left[p^B+1-\alpha-q+\frac{q}{2}\alpha\right]^-}$$

(iv) **Conclusion**

Thus, combining this result with the fact that π^A is concave on $-q-(1-\alpha) < p^A - p^B \leq 1-\alpha-q+\frac{q}{2}\alpha$, we conclude that on the interval $-q-(1-\alpha) \leq p^A - p^B \leq 1-\alpha$ the profit function of platform A under horizontal dominance has a unique maximum with respect to p^A and, therefore, is quasi-concave in p^A. We provide robustness with a simulation where we observe, for various combinations of the pair (α, q), a profit function with a quasi-concave shape.

B. proof under Vertical Dominance

In the case of vertical dominance we obtain that the profit function of platform A is given by:

$$\pi^{A|VD}\left(p^A, p^B\right) =$$

$$\begin{cases} 0, \; p^A - p^B > 1-\alpha; \; \text{(branch 5)} \\ \frac{p^A\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}\right]}{\alpha^2}, \; -(1+\alpha)+\frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha; \; \text{(branch 4)} \\ \frac{2p^A}{q-2\alpha}\left(p^B - p^A - \alpha\right), \; 1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q}; \; \text{(branch 3)} \\ \frac{p^A\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\,]}\right]}{\alpha^2}, \; -q-(1-\alpha) \leq p^A - p^B < 1+\alpha-q-\frac{2\alpha}{q}; \\ \hfill \text{(branch 2)} \\ 2p^A, p^A - p^B < -q-(1-\alpha). \; \text{(branch 1)} \end{cases}$$

(i) **Concave and linear segments of the profit function** *(branches 1-3 and 5)*

As explained in the case of horizontal dominance but, now, under vertical dominance, $D^{A|VD}\left(p^A,p^B\right)$ is concave and decreasing for $p^A-p^B\leq -(1+\alpha)+\frac{2\alpha}{q}$ and it is constant for $p^A-p^B>1-\alpha$. Since demand in those price domains is concave and non-increasing, the profit function of platform A is also a concave function of p^A in that price domain[17].

Note that for the concave segment $-q-(1-\alpha)\leq p^A-p^B<1+\alpha-q-\frac{2\alpha}{q}$ of the demand of platform A yields:

$$\begin{aligned}\frac{\partial D^{A|VD}\left(p^A,p^B\right)}{\partial p^A} &= \frac{1}{2\alpha}-\frac{\sqrt{q\left[q-2\alpha(1-\alpha+p^A-p^B+q)\ \right]}}{2\alpha\left[q-2\alpha(1-\alpha+p^A-p^B+q)\ \right]};\\ \frac{\partial^2 D^{A|VD}\left(p^A,p^B\right)}{\partial (p^A)^2} &= -\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}}<0.\end{aligned}$$

The profit function of platform A is linear at branch 1 and concave at branches 2 and 3. The first derivative of the profit function relatively to its price at branches 2 and 3 are, respectively, given by:

$$\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}=1+\frac{1}{\alpha}-\frac{q}{\alpha^2}+\frac{2p^A}{\alpha}+\frac{q}{\alpha}-\frac{p^B}{\alpha}+\frac{\sqrt{q[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]}}{\alpha^2}-\frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]};$$

$$\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}=-\frac{2\left(2p^A-p^B+\alpha\right)}{q-2\alpha}.$$

Evaluating the derivatives at $p^A=p^B+1+\alpha-q-\frac{2\alpha}{q}$, follows that the LHS derivative (using the profit function expression of branch 2) and the RHS derivative (using the profit function expression of branch 3) are given by:

$$\begin{aligned}\left.\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B+1+\alpha-q-\frac{2\alpha}{q}\right]^-} &= 3-\frac{4}{q}+\frac{q}{q-2\alpha}-\frac{2p^B}{q-2\alpha};\\ \left.\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B+1+\alpha-q-\frac{2\alpha}{q}\right]^+} &= 3-\frac{4}{q}+\frac{q}{q-2\alpha}-\frac{2p^B}{q-2\alpha}.\end{aligned}$$

Then, under vertical dominance, the LHS derivative and the RHS derivative at $p^A=p^B+1+\alpha-q-\frac{2\alpha}{q}$ are equal. Thus, the profit function is concave in the domain $-q-(1-\alpha)\leq p^A-p^B\leq-(1+\alpha)+\frac{2\alpha}{q}$.

[17]Note that since $\pi^A\left(p^A,p^B\right)=2p^AD^A(p^A,p^B)$, follows that $\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}=2D^A(p^A,p^B)+2p^A\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A}$. Thus, $\frac{\partial^2\pi^A\left(p^A,p^B\right)}{\partial (p^A)^2}=4\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A}+2p^A\frac{\partial^2 D^A\left(p^A,p^B\right)}{\partial (p^A)^2}$. Since the demand is (i) decreasing and (ii) concave at the considered segment follows that (i) $\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A}<0$ and (ii) $\frac{\partial^2 D^A\left(p^A,p^B\right)}{\partial (p^A)^2}$, respectively. Then, we obtain $\frac{\partial^2\pi^A\left(p^A,p^B\right)}{\partial (p^A)^2}<0$.

(ii) **Convex segment of the profit function** *(branch 4)*

By contrast, the only domain in which the profit function is not concave is:

$$-(1+\alpha)+\frac{2\alpha}{q}<p^A-p^B\leq 1-\alpha,$$

where the demand function $D^{A|VD}\left(p^A,p^B\right)$ is strictly convex, with:

$$\frac{\partial^2 D^{A|HD}\left(p^A,p^B\right)}{\partial (p^A)^2}=\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}}>0$$

for such values of $\left(p^A,p^B\right)$.

In that price domain, yields that the profit function is given by:

$$\pi_A\left(p^A,p^B\right)=2p^A D^A(p^A,p^B)=p^A\left\{\frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha^2}\right\},$$

The derivative relatively to price is given by expression (7). The second derivative of π^A relatively to p^A in the mentioned price domain is given by:

$$\frac{\partial^2\pi^A\left(p^A,p^B\right)}{\partial (p^A)^2}=\frac{2}{\alpha}+\frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]^2}-\frac{2\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]},$$

and the third derivative of the profit function with respect to their price is given by:

$$\frac{\partial^3\pi^A\left(p^A,p^B\right)}{\partial (p^A)^3}=\frac{3\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\left\{q-\alpha\left[2(1-\alpha)+2p^B-p^A\right]\ \right\}}{[q-2\alpha(1-\alpha+p^B-p^A)\ ]^3}.$$

Thus, it is straightforward that $\frac{\partial^3\pi^A\left(p^A,p^B\right)}{\partial (p^A)^3}>0$ under assumption 1, so that $\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}$ is strictly convex in p^A for the price domain $-(1-\alpha)+\frac{2\alpha}{q}<p^A-p^B\leq 1-\alpha$.

Also considering the behavior of $\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}$ in the neighborhood of $p^A=p^B+(1-\alpha)$, which corresponds to the upper bound of the convex segment of the demand, plugging $p^A=p^B+(1-\alpha)$ in equation (7), we obtain:

$$\left.\frac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=[p^B+(1-\alpha)]^-}=-\frac{\left[q+\alpha(1-\alpha+p^B)\ \right]\left(\sqrt{q^2}-q\right)}{q\alpha^2}=0.$$

Since the LHS derivative of π^A relatively to p^A is null at $p^A=\left[p^B+(1-\alpha)\right]^-$ and the second derivative of the profit function evaluated at $p^A=\left[p^B+(1-\alpha)\right]^-$ is given by:

$$\left.\frac{\partial^2\pi^A\left(p^A,p^B\right)}{\partial (p^A)^2}\right|_{p^A=[p^B+(1-\alpha)]^-}=\frac{p^B+1-\alpha}{q}>0,$$

it follows that the profit function $\pi^A\left(p^A, p^B\right)$ reaches to a minimum at the point $p^A = p^B + (1-\alpha)$ and, thus, $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 0$ has at most one solution in the domain $-(1+\alpha) + \frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha$.

(*iii*) **Sign of the derivatives at the kink between the linear and convex segment of the demand function**

Furthermore, in the neighborhood of $p^A = p^B - (1+\alpha) + \frac{2\alpha}{q}$ (the lower bound of the convex segment of the demand function), the RHS and the LHS derivatives are equal at this kink point of the profit function of platform A.

(*a*) To compute the RHS derivative, we plug $p^A = p^B - (1+\alpha) + \frac{2\alpha}{q}$ in equation (7) to obtain:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B-(1-\alpha)+\frac{2\alpha}{q}\right]^+} = \frac{4}{q-2\alpha} - \frac{2p^B}{q-2\alpha} + \frac{2\alpha}{q-2\alpha} - \frac{8\alpha}{q(q-2\alpha)}.$$

(*b*) To compute the LHS derivative note that, for $\left[p^B - (1+\alpha) + \frac{2\alpha}{q}\right]^-$, the profit of platform A is equal to:

$$\pi^A\left(p^A, p^B\right) = \frac{2p^A}{q-2\alpha}\left(p^B - p^A - \alpha\right).$$

Then, the derivative is given by:

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = \frac{2p^B}{q-2\alpha} - \frac{4p^A}{q-2\alpha} - \frac{2\alpha}{q-2\alpha}.$$

Evaluating the derivative at $p^A = p^B - (1+\alpha) + \frac{2\alpha}{q}$, we obtain that the LHS derivative in the neighborhood of $p^A = \left[p^B - (1+\alpha) + \frac{2\alpha}{q}\right]^-$ is given by:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B-(1-\alpha)+\frac{2\alpha}{q}\right]^-} = \frac{4}{q-2\alpha} - \frac{2p^B}{q-2\alpha} + \frac{2\alpha}{q-2\alpha} - \frac{8\alpha}{q(q-2\alpha)}.$$

Then, both derivatives have the same sign and are equal.

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B-(1-\alpha)+\frac{2\alpha}{q}\right]^-} = \left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B-(1-\alpha)+\frac{2\alpha}{q}\right]^+},$$

such that the profit function is continuous and decreasing.

(*iv*) **Conclusion**

Thus, combining this result with the fact that π^A is concave on $-q-(1-\alpha) \leq p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q}$, we conclude that on the interval $-q-(1-\alpha) \leq p^A - p^B \leq 1-\alpha$ the profit function of platform A under vertical dominance is quasi-concave and it has a unique maximum with respect to p^A. We provide robustness with a simulation where we observe, for various combinations of pairs (α, q), a profit function with a quasi-concave shape.

The proof can be repeated, *mutatis mutandis*, under vertical and horizontal dominance for platform B. Let us now consider the profit of platform B :

$$\pi^B\left(p^A, p^B\right) = 2p^B D^B(p^A, p^B),$$

with $D^B(p^A, p^B) = D^{B|VD}\left(p^A, p^B\right)$ under vertical dominance, and $D^B(p^A, p^B) = D^{B|HD}\left(p^A, p^B\right)$ under horizontal dominance. Recall that the demand of platform i is a continuous and non-increasing function of p^i, both under horizontal and vertical dominance.

C. proof under Horizontal Dominance

In the case of horizontal dominance we obtain that the profit function of platform B is given by:

$\pi^{B|HD}\left(p^A, p^B\right) =$

$$
\begin{cases}
2p^B, \; p^A - p^B > 1-\alpha; \text{ (branch 5)} \\
2p^B\left(1 - \frac{\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}\right]}{2\alpha^2}\right), \quad 1-\alpha-q+\frac{q\alpha}{2} < p^A - p^B \leq 1-\alpha; \\
\hfill \text{(branch 4)} \\
2p^B\left\{1 - \frac{1}{2}\left[1 + \frac{p^B-p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right]\right\}, \quad -(1-\alpha) - \frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha-q+\frac{q\alpha}{2}; \\
\hfill \text{(branch 3)} \\
2p^B\left\{1 - \frac{\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\,]}\right]}{2\alpha^2}\right\}, \quad -q-(1-\alpha) \leq p^A - p^B < \\
\hfill < -(1-\alpha) - \frac{q\alpha}{2}; \text{ (branch 2)} \\
0, \; p^A - p^B < -q-(1-\alpha). \text{ (branch 1)}
\end{cases}
$$

(i) **Concave and linear segments of the profit function** (*branches 3-5 and 1*)

Furthermore, in the case of horizontal dominance, $D^{B|HD}\left(p^A,p^B\right)$ is concave and decreasing for $-(1-\alpha) - \frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha$ and it is constant for $p^A - p^B > 1-\alpha$ and $p^A - p^B < -q-(1-\alpha)$. Since demand in those price domains is concave and non-increasing, the profit function of platform B is also a concave function of p^B in that price domain[18]. Note that for the concave segment $1-\alpha-q+\frac{q\alpha}{2} < p^A - p^B \leq 1-\alpha$ of the demand of platform B yields:

$$
\begin{aligned}
\frac{\partial D^{B|HD}\left(p^A,p^B\right)}{\partial p^B} &= -\frac{-\alpha+\frac{q\alpha}{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}}{2\alpha^2}; \\
\frac{\partial^2 D^{B|HD}\left(p^A,p^B\right)}{\partial \left(p^B\right)^2} &= -\frac{1}{2q^{\frac{1}{2}}\,[q-2\alpha(p^B-p^A+1-\alpha)\,]^{\frac{3}{2}}} < 0.
\end{aligned}
$$

The profit function of platform B is linear at branch 5 and concave at branches 3 and 4. The first derivative of the profit function relatively to its price at branches 4 and 3 are,

[18] Note that since $\pi^B\left(p^A,p^B\right) = 2p^B D^B(p^A,p^B)$, follows that $\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = 2D^B(p^A,p^B) + 2p^B\frac{\partial D^B\left(p^A,p^B\right)}{\partial p^B}$. Thus, $\frac{\partial^2 \pi^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^2} = 4\frac{\partial D^B\left(p^A,p^B\right)}{\partial p^B} + 2p^B\frac{\partial^2 D^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^2}$. Since the demand is (i) decreasing and (ii) concave at the considered segment follows that (i) $\frac{\partial D^B\left(p^A,p^B\right)}{\partial p^B} < 0$ and (ii) $\frac{\partial^2 D^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^2}$, respectively. Then, we obtain $\frac{\partial^2 \pi^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^2} < 0$.

respectively, given by:

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = 1-\frac{q}{\alpha^2}+\frac{1}{\alpha}-\frac{p^A}{\alpha}+\frac{2p^B}{\alpha}+\frac{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha^2}-\frac{p^B\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]};$$

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = \frac{2(1-\alpha)+q+2p^A-4p^B}{2(1-\alpha)}.$$

Evaluating the derivatives at the kink $p^B = p^A-(1-\alpha)+q-\frac{q\alpha}{2}$, follows that the LHS derivative (using the profit function expression of branch 4) and the RHS derivative (using the profit function expression of branch 3) are given by:

$$\begin{aligned}\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}\right|_{p^B=\left[p^A-(1-\alpha)+q-\frac{q\alpha}{2}\right]^-} &= 3-q-\frac{q+2p^A}{2(1-\alpha)};\\ \left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A-(1-\alpha)+q-\frac{q\alpha}{2}\right]^+} &= 3-q-\frac{q+2p^A}{2(1-\alpha)}.\end{aligned}$$

In the frontier between the two concave segments of the profit function, the following inequality must hold:

$$\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A-(1-\alpha)+q-\frac{q\alpha}{2}\right]^-} \geq \left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A-(1-\alpha)+q-\frac{q\alpha}{2}\right]^+}.$$

We verify that this inequality is verified, since the LHS derivative is equal to the RHS derivative at $p^B = p^A-(1-\alpha)+q-\frac{q\alpha}{2}$.

(*ii*) **Convex segment of the profit function** (*branch 2*)

By contrast, the only domain in which the profit function is not concave is:

$$-q-(1-\alpha) \leq p^A-p^B < -(1-\alpha)-\frac{q\alpha}{2},$$

where the demand function $D^{B|HD}\left(p^A,p^B\right)$ is strictly convex, with:

$$\frac{\partial^2 D^{B|HD}\left(p^A,p^B\right)}{\partial \left(p^B\right)^2} = \frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}} > 0$$

for such values of $\left(p^A,p^B\right)$.

In that price domain, yields that the profit function is given by:

$$\pi^B\left(p^A,p^B\right) = 2p^B D^B(p^A,p^B) = 2p^B\left\{1-\frac{\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{2\alpha^2}\right\},$$

The derivative relatively to price is given by:

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = \frac{q-q\alpha-\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}-\alpha\left(1-\alpha-2p^B+p^A+\frac{qp^B}{\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}\right)}{\alpha^2} \tag{9}$$

Alternatively, expression (9) can be re-written as:

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = 1+\frac{q}{\alpha^2}-\frac{1}{\alpha}-\frac{p^A}{\alpha}-\frac{q}{\alpha}+\frac{2p^B}{\alpha}-\frac{p^B\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}{\alpha[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}-\frac{\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}{\alpha^2}. \tag{10}$$

The second derivative of π^B relatively to p^B in the mentioned price domain is given by:

$$\frac{\partial^2 \pi^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^2} = \frac{2+\frac{q^2p^B\alpha}{\left(q-2\alpha(1-\alpha-p^B+p^A+q)\ \right)^{\frac{3}{2}}}-\frac{2q}{\sqrt{q\left(q-2\alpha(1-\alpha-p^B+p^A+q)\ \right)}}}{\alpha^2},$$

and the third derivative of the profit function with respect to their price is given by:

$$\frac{\partial^3 \pi^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^3} = \frac{3q^3\left[q-2\alpha\left(1-\alpha-\frac{p^B}{2}+p^A+q\right)\right]}{\left[q\left(q-2\alpha(1-\alpha-p^B+p^A+q)\ \right)\right]^{\frac{5}{2}}}.$$

Thus, it is straightforward that $\frac{\partial^3 \pi^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^3} > 0$ under assumption 1, so that $\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}$ is strictly convex in p^B for the price domain $-q-(1-\alpha) \leq p^A-p^B < -(1-\alpha)-\frac{q\alpha}{2}$.

Also considering the behavior of $\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}$ in the neighborhood of $p^B = p^A+(1-\alpha)+q$, which corresponds to the upper bound of the convex segment of the demand, plugging $p^B = p^A+(1-\alpha)+q$ in equation (9), we obtain:

$$\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}\right|_{p^B=[p^A+(1-\alpha)+q]^-} = -\frac{\left(-q+\sqrt{q^2}\right)\left[q+(1+p^A+q)\alpha-\alpha^2\right]}{q\alpha^2} = 0.$$

Since the LHS derivative of π^B relatively to p^B is null at $p^B = \left[p^A+(1-\alpha)+q\right]^-$ and the second derivative of the profit function evaluated at $p^B = \left[p^A+(1-\alpha)+q\right]^-$ is given by:

$$\left.\frac{\partial^2 \pi^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^2}\right|_{p^B=[p^A+(1-\alpha)+q]^-} = \frac{1-\alpha+p^A+q}{q} > 0,$$

it follows that the profit function $\pi^B\left(p^A,p^B\right)$ reaches to a minimum at the point $p^B = p^A+(1-\alpha)+q$ and, thus, $\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = 0$ has at most one solution in the domain $-q-(1-\alpha) \leq p^A-p^B < -(1-\alpha)-\frac{q\alpha}{2}$.

(iii) **Sign of the derivatives at the kink between the linear and convex segment of the demand function**

Furthermore, in the neighborhood of $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$, the lower bound of the convex segment of the demand function, the RHS and the LHS derivatives at the kink point $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$ of the profit function of platform B are equal.

(a) To compute the RHS derivative, we plug $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$ in equation (9) to obtain:

$$\frac{\partial \pi^B\left(p^A, p^B\right)}{\partial (p^B)}\Bigg|_{p^B=\left[p^A+1-\alpha+\frac{q}{2}\alpha\right]^+} = -1 + q - \frac{2p^A + q}{2(1-\alpha)}.$$

(b) To compute the LHS derivative note that, for $p^B = \left[p^A + 1 - \alpha + \frac{q}{2}\alpha\right]^-$, the profit of platform B is equal to:

$$\pi^B\left(p^A, p^B\right) = 2p^B\left\{1 - \frac{1}{2}\left[1 + \frac{p^B - p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right]\right\}.$$

Then, the derivative is given by:

$$\frac{\partial \pi^B\left(p^A, p^B\right)}{\partial p^B} = \frac{2 - 2\alpha + q - 4p^B + 2p^A}{2(1-\alpha)}.$$

Evaluating the derivative at $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$, we obtain the LHS derivative in the neighborhood of $p^B = \left[p^A + 1 - \alpha + \frac{q}{2}\alpha\right]^-$:

$$\frac{\partial \pi^B\left(p^A, p^B\right)}{\partial p^B}\rceil_{p^B=\left[p^A+1-\alpha+\frac{q}{2}\alpha\right]^-} = -1 + q - \frac{2p^A + q}{2(1-\alpha)}.$$

Thus, the sign of $\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial p^A}$ is the same on both sides of $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$ and both derivatives are equal:

$$\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\rceil_{p^A=\left[p^B+1-\alpha+\frac{q}{2}\alpha\right]^+} = \frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\rceil_{p^A=\left[p^B+1-\alpha+\frac{q}{2}\alpha\right]^-}$$

(iv) **Conclusion**

Thus, combining this result with the fact that π^B is concave on $1 - \alpha - q + \frac{q\alpha}{2} \leq p^A - p^B \leq 1 - \alpha$, we conclude that the profit function of platform B under horizontal dominance is quasi-concave in p^B and, therefore, has a unique maximum with respect to p^B.

D. proof under Vertical Dominance

In the case of vertical dominance we obtain that the profit function of platform B is given by:

$$\pi^{B|VD}\left(p^A,p^B\right) =$$

$$\begin{cases} 2p^B,\ p^A-p^B>1-\alpha;\ \text{(branch 5)} \\ 2p^B\left\{1-\frac{\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\right]}{2\alpha^2}\right\},\quad -(1+\alpha)+\frac{2\alpha}{q}<p^A-p^B\le 1-\alpha;\ \text{(branch 4)} \\ 2p^B\left[1-\frac{1}{q-2\alpha}\left(p^B-p^A-\alpha\right)\right],\ 1+\alpha-q-\frac{2\alpha}{q}\le p^A-p^B\le -(1+\alpha)+\frac{2\alpha}{q};\ \text{(branch 3)} \\ 2p^B\left\{1-\frac{\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{2\alpha^2}\right\},\quad -q-(1-\alpha)\le p^A-p^B< \\ \qquad < 1+\alpha-q-\frac{2\alpha}{q};\ \text{(branch 2)} \\ 0,\ p^A-p^B<-q-(1-\alpha).\ \text{(branch 1)} \end{cases}$$

(i) **Concave and linear segments of the profit function** *(branches 3-5 and 1)*

As explained in the case of horizontal dominance but, now, under vertical dominance, $D^{B|VD}\left(p^A,p^B\right)$ is concave and decreasing for $1+\alpha-q-\frac{2\alpha}{q}\le p^A-p^B\le 1-\alpha$ and it is constant for $p^A-p^B<-q-(1-\alpha)$. Since demand in those price domains is concave and non-increasing, the profit function of platform B is also a concave function of p^B in that price domain. Note that for the concave segment $-(1+\alpha)+\frac{2\alpha}{q}<p^A-p^B\le 1-\alpha$ of the demand of platform B yields:

$$\begin{aligned} \frac{\partial D^{B|VD}\left(p^A,p^B\right)}{\partial p^B} &= -\frac{-\alpha+\frac{q\alpha}{\sqrt{q\left[q-2\alpha(1-\alpha+p^B-p^A)\ \right]}}}{2\alpha^2}; \\ \frac{\partial^2 D^{B|VD}\left(p^A,p^B\right)}{\partial \left(p^B\right)^2} &= -\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha(p^B-p^A+1-\alpha)\ \right]^{\frac{3}{2}}}<0. \end{aligned}$$

The profit function of platform B is linear at branch 5 and concave at branches 3 and 4. The first derivative of the profit function relatively to its price at branches 4 and 3 are, respectively, given by:

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = 1 - \frac{q}{\alpha^2} + \frac{1}{\alpha} - \frac{p^A}{\alpha} + \frac{2p^B}{\alpha} + \frac{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha^2} - \frac{p^B\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]};;$$

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = \frac{2\left(q-\alpha+p^A-2p^B\ \right)}{q-2\alpha}.$$

Evaluating the derivatives at $p^B = p^A + 1 + \alpha - \frac{2\alpha}{q}$, follows that the LHS derivative (using the profit function expression of branch 4) and the RHS derivative (using the profit function expression of branch 3) are given by:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A+1+\alpha-\frac{2\alpha}{q}\right]^-} = 3 - \frac{4}{q} - \frac{2p^A+q}{q-2\alpha};$$

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A+1+\alpha-\frac{2\alpha}{q}\right]^+} = 3 - \frac{4}{q} - \frac{2p^A+q}{q-2\alpha}.$$

Then, under vertical dominance, the LHS derivative and the RHS derivative at $p^B = p^A + 1 + \alpha - \frac{2\alpha}{q}$ are equal. Thus, the profit function is strictly concave in the domain $1 + \alpha - q - \frac{2\alpha}{q} \leq p^A - p^B \leq 1 - \alpha$.

(ii) **Convex segment of the profit function** *(branch 2)*

By contrast, the only domain in which the profit function is not concave is:

$$-q - (1-\alpha) \leq p^A - p^B < 1 + \alpha - q - \frac{2\alpha}{q},$$

where the demand function $D^{B|VD}\left(p^A,p^B\right)$ is strictly convex, with:

$$\frac{\partial^2 D^{B|VD}\left(p^A,p^B\right)}{\partial \left(p^B\right)^2} = \frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}} > 0$$

for such values of $\left(p^A,p^B\right)$.

In that price domain, yields that the profit function is given by:

$$\pi^B\left(p^A,p^B\right) = 2p^B D^B(p^A,p^B) = 2p^B\left\{1 - \frac{\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{2\alpha^2}\right\},$$

The derivative relatively to price is given by expression (9). The second derivative of π^B relatively to p^B in the mentioned price domain is given by:

$$\frac{\partial^2 \pi^B\left(p^A,p^B\right)}{\partial \left(p^B\right)^2} = \frac{2+\frac{q^2p^B\alpha}{\left[q-2\alpha(1-\alpha-p^B+p^A+q)\ \right]^{\frac{3}{2}}}-\frac{2q}{\sqrt{q\left[q-2\alpha(1-\alpha-p^B+p^A+q)\ \right]}}}{\alpha^2},$$

and the third derivative of the profit function with respect to their price is given by:

$$\frac{\partial^3 \pi^B\left(p^A,p^B\right)}{\partial (p^B)^3} = \frac{3q^3\left[q - 2\alpha\left(1-\alpha-\frac{p^B}{2}+p^A+q\right)\right]}{\{q\left[q-2\alpha(1-\alpha-p^B+p^A+q)\ \right]\}^{\frac{5}{2}}}.$$

Thus, it is straightforward that $\frac{\partial^3 \pi^B\left(p^A,p^B\right)}{\partial (p^B)^3} > 0$ under assumption 1, so that $\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}$ is strictly convex in p^B for the price domain $-q-(1-\alpha) \leq p^A - p^B < 1+\alpha-q-\frac{2\alpha}{q}$.

Also considering the behavior of $\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}$ in the neighborhood of $p^B = p^A + (1-\alpha) + q$, which corresponds to the upper bound of the convex segment of the demand, plugging $p^B = p^A + (1-\alpha) + q$ in equation (9), we obtain:

$$\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}\right|_{p^B=[p^A+(1-\alpha)+q]^-} = -\frac{\left(-q+\sqrt{q^2}\right)\left[q+(1+p^A+q)\alpha-\alpha^2\right]}{q\alpha^2} = 0.$$

Since the LHS derivative of π^B relatively to p^B is null at $p^B = \left[p^A + (1-\alpha) + q\right]^-$ and the second derivative of the profit function evaluated at $p^B = \left[p^A + (1-\alpha) + q\right]^-$ is given by:

$$\left.\frac{\partial^2 \pi^B\left(p^A,p^B\right)}{\partial (p^B)^2}\right|_{p^B=[p^A+(1-\alpha)+q]^-} = \frac{1-\alpha+p^A+q}{q} > 0,$$

it follows that the profit function $\pi^B\left(p^A,p^B\right)$ reaches to a minimum at the point $p^B = p^A + (1-\alpha) + q$ and, thus, $\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = 0$ has at most one solution in the domain $-q-(1-\alpha) \leq p^A - p^B < 1+\alpha-q-\frac{2\alpha}{q}$.

(iii) **Sign of the derivatives at the kink between the linear and convex segment of the demand function**

Furthermore, in the neighborhood of $p^B = p^A - (1+\alpha) + q + \frac{2\alpha}{q}$ (the lower bound of the convex segment of the demand function), the RHS and the LHS derivatives are equal at this kink point of the profit function of platform B.

(a) To compute the RHS derivative, we plug $p^B = p^A - (1+\alpha) + q + \frac{2\alpha}{q}$ in equation (9) to obtain:

$$\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial (p^B)}\right|_{p^B=\left[p^A-(1+\alpha)+q+\frac{2\alpha}{q}\right]^+} = -1 + \frac{4}{q} - \frac{2p^A+q}{q-2\alpha}.$$

(b) To compute the LHS derivative note that, for $p^B = \left[p^A - (1+\alpha) + q + \frac{2\alpha}{q}\right]^-$, the profit of platform B is equal to:

$$\pi^B\left(p^A, p^B\right) = 2p^B\left[1 - \frac{1}{q-2\alpha}\left(p^B - p^A - \alpha\right)\right].$$

Then, the derivative is given by:

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = \frac{2(p^A + q - 2p^B - \alpha)}{q-2\alpha}.$$

Evaluating the derivative at $p^B = p^A - (1+\alpha) + q + \frac{2\alpha}{q}$, we obtain that the LHS derivative in the neighborhood of $p^B = \left[p^A - (1+\alpha) + q + \frac{2\alpha}{q}\right]^-$ is given by:

$$\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}\right|_{p^B=\left[p^A-(1+\alpha)+q+\frac{2\alpha}{q}\right]^-} = -1 + \frac{4}{q} - \frac{2p^A + q}{q-2\alpha}.$$

Then, both derivatives have the same sign and are equal:

$$\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}\right|_{p^B=\left[p^A-(1+\alpha)+q+\frac{2\alpha}{q}\right]^-} = \left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}\right|_{p^B=\left[p^A-(1+\alpha)+q+\frac{2\alpha}{q}\right]^+}.$$

(iv) **Conclusion**

Thus, combining this result with the fact that π^B is concave on $1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq 1-\alpha$, we conclude that the profit function of platform B under vertical dominance is quasi-concave and it has a unique maximum with respect to p^B. □

Proof of Proposition 2

Consider now the case of horizontal dominance, arising when $q < 2$. When:

$$-(1-\alpha) - \frac{q\alpha}{2} \leq p^A - p^B \leq 1 - \alpha - q + \frac{q\alpha}{2},$$

the profit of platform A is equal to:

$$2p^A\left\{\frac{1}{2}\left[1 + \frac{p^B - p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right]\right\}$$

while the profit of platform B is given by:

$$2p^B\left\{1 - \frac{1}{2}\left[1 + \frac{p^B - p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right]\right\}.$$

Solving the corresponding system of first order conditions, we obtain the equilibrium price candidate:

$$p^{A^*} = 1 - \alpha - \tfrac{q}{6};\ p^{B^*} = 1 - \alpha + \tfrac{q}{6},$$

and equilibrium market shares:

$$D^{A^*} = \tfrac{1}{2}\left[1 - \tfrac{q}{6(1-\alpha)}\right],\ D^{B^*} = \tfrac{1}{2}\left[1 + \tfrac{q}{6(1-\alpha)}\right].$$

The equilibrium candidate requires:

$$D^{A^*} \geq 0 \Leftrightarrow q < 6(1-\alpha),$$

which holds under horizontal dominance accordingly to assumption 1.

It only remains to verify if the equilibrium price candidate verifies the price domain condition:

$$-(1-\alpha) - \tfrac{q\alpha}{2} \leq p^A - p^B \leq 1 - \alpha - q + \tfrac{q\alpha}{2},$$

which holds for the intersection of the below system of inequalities. Computing, we obtain:

$$\begin{cases} -(1-\alpha) - \frac{q\alpha}{2} \leq p^{A*} - p^{B*}; \\ p^{A*} - p^{B*} \leq 1 - \alpha - q + \frac{q\alpha}{2}. \end{cases} \Leftrightarrow \begin{cases} -(1-\alpha) - \frac{q\alpha}{2} \leq -\frac{q}{3}; \\ -\frac{q}{3} \leq 1 - \alpha - q + \frac{q\alpha}{2}. \end{cases} \Leftrightarrow$$
$$\Leftrightarrow q \leq \tfrac{6(1-\alpha)}{2-3\alpha} \cap q \leq \tfrac{6(1-\alpha)}{4-3\alpha}. \Leftrightarrow q < \tfrac{6(1-\alpha)}{4-3\alpha},$$

as described by the yellow region exposed in figure 6.

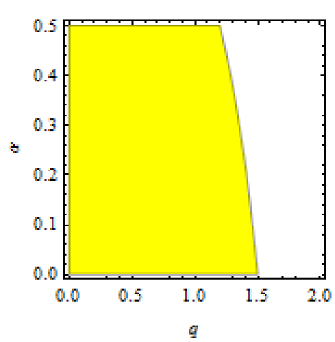

Figure 6

yielding Proposition 2 that is, now, fully characterized. □

Proof of Proposition 3

As the profit function is quasi-concave, for each type of dominance, the equilibrium prices can be obtained from the first order conditions:

$$\frac{\partial \pi^i\left(p^A,p^B\right)}{\partial p^i} = 0,\ \ i \in \{A, B\}\,.$$

Recall that, we concentrate on interior solutions in which both platforms are active. Thus, we exclude the pairs of prices $\left(p^A, p^B\right)$ such that:

$$p^A - p^B > 1 - \alpha \vee p^A - p^B < -q - (1 - \alpha).$$

Let us consider first, the case of vertical dominance, occurring when $q > 2$.

Focusing only on the linear segment of the profit function, when:

$$1 + \alpha - q - \frac{2\alpha}{q} \leq p^A - p^B \leq -(1 - \alpha) + \frac{2\alpha}{q},$$

the profit of platform A is equal to:

$$2p^A \left[\frac{1}{q - 2\alpha}\left(p^B - p^A - \alpha\right)\right]$$

while the profit of platform B is given by:

$$2p^B \left[1 - \frac{1}{q - 2\alpha}\left(p^B - p^A - \alpha\right)\right].$$

Solving the corresponding system of first order conditions, we obtain the equilibrium prices:

$$p^{A^*} = \tfrac{q-\alpha}{3}; p^{B^*} = \tfrac{2(q-\alpha)}{3},$$

and the equilibrium market shares:

$$D^{A^*} = \tfrac{1}{3} - \tfrac{2\alpha}{3(q-2\alpha)}, D^{B^*} = \tfrac{2}{3} + \tfrac{2\alpha}{3(q-2\alpha)}.$$

The equilibrium candidate requires:

$$D^{A^*} \geq 0 \Leftrightarrow q > 4\alpha,$$

which holds under vertical dominance accordingly to assumption 1.

Given the quasi-concavity of the profit function, we obtain that this equilibrium price candidate is always an equilibrium, as long as:

$$1+\alpha-q-\tfrac{2\alpha}{q}\leq p^{A}-p^{B}\leq-(1-\alpha)+\tfrac{2\alpha}{q}.$$

This leads to the following result:

$$\begin{cases}1+\alpha-q-\frac{2\alpha}{q}\leq p^{A*}-p^{B*};\\ p^{A*}-p^{B*}\leq-(1-\alpha)+\frac{2\alpha}{q}.\end{cases}\Leftrightarrow\begin{cases}1+\alpha-q-\frac{2\alpha}{q}\leq-\frac{q-\alpha}{3};\\ -\frac{q-\alpha}{3}\leq-(1-\alpha)+\frac{2\alpha}{q}.\end{cases}$$

$$\Leftrightarrow\begin{cases}1+\frac{2}{3}\alpha-\frac{2}{3}q-\frac{2\alpha}{q}\leq 0;\\ 1-\frac{2}{3}\alpha-\frac{1}{3}q-\frac{2\alpha}{q}\leq 0.\end{cases}\Leftrightarrow\begin{cases}-2q^{2}+q(3+2\alpha)-6\alpha\leq 0;\\ -q^{2}+q(3-2\alpha)-6\alpha\leq 0.\end{cases}$$

Taking into consideration the definition of vertical dominance ($q>2$), the equilibrium price candidate is always an equilibrium for:

$$q>2\cap-2q^{2}+q(3+2\alpha)-6\alpha\leq 0\cap-q^{2}+q(3-2\alpha)-6\alpha\leq 0,$$

which is described by the blue region in the figure 7.

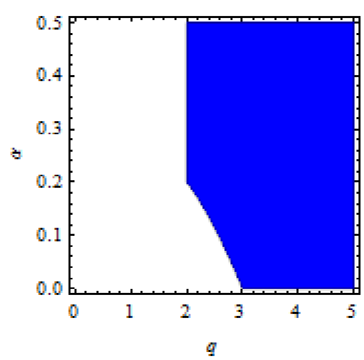

Figure 7

The roots of both polynomials are given by:

$$\begin{cases}-2q^{2}+q(3+2\alpha)-6\alpha=0\\ -q^{2}+q(3-2\alpha)-6\alpha=0\end{cases}\Leftrightarrow\begin{cases}q_{cr1*}=\frac{1}{4}\left(3+2\alpha\pm\sqrt{9-4\alpha(9-\alpha)}\right);\\ q_{cr2*}=\frac{1}{2}\left(3-2\alpha\pm\sqrt{9-4\alpha(9-\alpha)}\right).\end{cases}$$

Then, the equilibrium price candidate is an equilibrium for:

$$q>2\cap q>\underline{q},$$

with $\underline{q} = \frac{1}{2}\left(3 - 2\alpha + \sqrt{9 - 4\alpha(9-\alpha)}\right)$. □

Proof of Lemma 5

Under the region of horizontal dominance $q < \frac{6(1-\alpha)}{4-3\alpha}$, the derivatives with respect to the degree of vertical differentiation are given by:

$$\frac{\partial p^{A*}}{\partial q} = -\frac{1}{6} < 0 \text{ and } \frac{\partial p^{B*}}{\partial q} = \frac{1}{6} > 0;$$
$$\frac{\partial D^{A*}}{\partial q} = -\frac{1}{12(1-\alpha)} < 0 \text{ and } \frac{\partial D^{B*}}{\partial q} = \frac{1}{12(1-\alpha)} > 0;$$
$$\frac{\partial \pi^{A*}}{\partial q} = \frac{q-6(1-\alpha)}{18} < 0 \text{ for } q < \frac{3}{2}(1-\alpha) \text{ and } \frac{\partial \pi^{B*}}{\partial q} = \frac{q+6(1-\alpha)}{18} > 0.$$

Since for $q \in (0,2)$ yields that $\frac{6(1-\alpha)}{4-3\alpha} > \frac{3}{2}(1-\alpha)$, (cf. figure 8 where the green region corresponds to $q < \frac{3}{2}(1-\alpha)$ and the yellow region corresponds to $\frac{3}{2}(1-\alpha) < q < \frac{6(1-\alpha)}{4-3\alpha}$), it follows that:

$$(i) \text{ for } q \in \left(0, \frac{3}{2}(1-\alpha)\right) : \frac{\partial \pi^{A*}}{\partial q} < 0; \frac{\partial \pi^{B*}}{\partial q} > 0;$$
$$(ii) \text{ for } q \in \left(\frac{3}{2}(1-\alpha), \frac{6(1-\alpha)}{4-3\alpha}\right) : \frac{\partial \pi^{A*}}{\partial q} > 0; ; \frac{\partial \pi^{B*}}{\partial q} > 0.$$

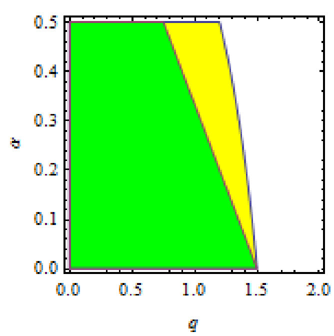

Figure 8

Under vertical dominance, $q > 2 \cap q > \frac{1}{2}\left(3 - 2\alpha + \sqrt{9 - 4\alpha(9-\alpha)}\right)$, yields:

$$\frac{\partial p^{A*}}{\partial q} = \frac{1}{3} > 0 \text{ and } \frac{\partial p^{B*}}{\partial q} = \frac{2}{3} > 0;$$
$$\frac{\partial D^{A*}}{\partial q} = \frac{2\alpha}{3(q-2\alpha)^2} > 0 \text{ and } \frac{\partial D^{B*}}{\partial q} = -\frac{2\alpha}{3(q-2\alpha)^2} < 0;$$

The derivative of the profit of the low-quality platform with respect to q is given by:

$$\frac{\partial \pi^{A^*}}{\partial q} = \frac{2(q-4\alpha)}{9(q-2\alpha)} + \frac{4\alpha(q-\alpha)}{9(q-2\alpha)^2} = \frac{2q^2-8q\alpha+12\alpha^2}{9q^2-36q\alpha+36\alpha^2} > 0.$$

Since both equilibrium prices and equilibrium market shares are increasing in q, the result is trivial.

On the high-quality platform B it follows that:

$$\frac{\partial \pi^{B^*}}{\partial q} = \frac{8q}{9(q-2\alpha)} - \frac{8\alpha(q-\alpha)}{9(q-2\alpha)^2} = \frac{8q^2-32q\alpha+24\alpha^2}{9(q-2\alpha)^2} \lesseqgtr 0.$$

The roots of the polynomial $8q^2 - 32q\alpha + 24\alpha^2$ are given by $q_{CR1} = \alpha$ and $q_{CR2} = 3\alpha$. By assumption 1: $0 \leq \alpha < \frac{1}{2}$. Then, it follows that: $q_{CR1} < 2 \cap q_{CR2} < 2, \forall \alpha$. Since under vertical dominance, $q > 2$, it is immediate that $\frac{\partial \pi^{B^*}}{\partial q} > 0$. □

Proof of Lemma 6

Under the region of horizontal dominance $q < \frac{6(1-\alpha)}{4-3\alpha}$, the derivatives of the outcomes with respect to the intensity of the network effect are given by:

$$\frac{\partial p^{A^*}}{\partial \alpha} = -1 < 0 \text{ and } \frac{\partial p^{B^*}}{\partial \alpha} = -1 < 0;$$
$$\frac{\partial D^{A^*}}{\partial \alpha} = -\frac{q}{12(1-\alpha)^2} < 0 \text{ and } \frac{\partial D^{B^*}}{\partial \alpha} = \frac{q}{12(1-\alpha)^2} > 0;$$
$$\frac{\partial \pi^{A^*}}{\partial \alpha} = -1 + \frac{q^2}{36(1-\alpha)^2} \text{ and } \frac{\partial \pi^{B^*}}{\partial \alpha} = -1 + \frac{q^2}{36(1-\alpha)^2}.$$

The roots of the polynomial $q^2 - 36(1-\alpha)^2$ are given by $q_{CR} = \pm 6(1-\alpha)$. Then, both derivatives are negative for $0 < q < \frac{6(1-\alpha)}{4-3\alpha}$.

Under the region of vertical dominance $q > 2 \cap q > \frac{1}{2}\left(3 - 2\alpha + \sqrt{9 - 4\alpha(9-\alpha)}\right)$, follows:

$$\frac{\partial p^{A^*}}{\partial \alpha} = -\frac{1}{3} < 0 \text{ and } \frac{\partial p^{B^*}}{\partial \alpha} = -\frac{2}{3} < 0;$$
$$\frac{\partial D^{A^*}}{\partial \alpha} = -\frac{2q}{3(q-2\alpha)^2} < 0 \text{ and } \frac{\partial D^{B^*}}{\partial \alpha} = \frac{2q}{3(q-2\alpha)^2} > 0;$$

The derivative of the profit of the low-quality platform with respect to α is given by:

$$\frac{\partial \pi^{A^*}}{\partial \alpha} = -\frac{6q^2 - 16q\alpha + 16\alpha^2}{9(q-2\alpha)^2} < 0.$$

Notice that for the polynomial $-(6q^2 - 16q\alpha + 16\alpha^2)$ yields no roots on the real space. However, since both equilibrium prices and equilibrium market shares are decreasing in α, the result is trivial.

On the high-quality platform B, follows that:

$$\frac{\partial \pi^{B^*}}{\partial \alpha} = \frac{16\alpha(q-\alpha)}{9(q-2\alpha)^2} \lesseqgtr 0.$$

The roots of the polynomial $16\alpha(q-\alpha)$ are given by $\alpha_{CR1} = 0$ and $\alpha_{CR2} = q$. By definition, vertical dominance requires: $q > 2$. Since, by assumption, $\alpha \in \left(0, \frac{1}{2}\right)$, it follows that $\frac{\partial \pi^{B^*}}{\partial \alpha} > 0$ for $\alpha \in \left(0, \frac{1}{2}\right)$. Lemma 6 is, now, straightforward. □

7.3 Appendix C - Intermediate equilibrium details

As mentioned in the paper, the white region corresponds to the equilibrium candidate occurring under the circumstances described in figure 1(b). In this situation, the equilibrium occurs in the strictly convex segment of D^A and in the strictly concave segment of D^B. The profit of platform A is given by:

$$\pi^A = \frac{p^A\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}\right]}{\alpha^2},$$

while the profit of platform B is given by:

$$\pi^B = 2p^B\left\{1-\frac{\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}\right]}{2\alpha^2}\right\}.$$

Given the quasi-concavity of the profit function, an equilibrium price candidate is always an equilibrium as long as the price domain condition is verified. Under horizontal dominance, the equilibrium is verified for:

$$1-\alpha-q+\tfrac{q\alpha}{2} < p^{A*}-p^{B*} \leq 1-\alpha$$

and under vertical dominance, the equilibrium is verified for:

$$-(1+\alpha)+\tfrac{2\alpha}{q} < p^{A*}-p^{B*} \leq 1-\alpha.$$

The best reply functions of platform A given the price charged by platform B are given by:

$$\begin{cases} p^A\left(p^B\right) = 1-\alpha+p^B; \\ p^A\left(p^B\right) = \frac{-3q+8\alpha\left(1-\alpha+p^B\right)-\sqrt{q}\sqrt{9q-16\alpha(1-\alpha+p^B)}}{16\alpha}; \\ p^A\left(p^B\right) = \frac{-3q+8\alpha\left(1-\alpha+p^B\right)+\sqrt{q}\sqrt{9q-16\alpha(1-\alpha+p^B)}}{16\alpha}. \end{cases}$$

Given an increment on the price of the high-quality platform, the low-quality platform reacts by also increasing its price. Analytically:

$$\frac{dp^A\left(p^B\right)}{dp^B} = 1 \cup \frac{dp^A\left(p^B\right)}{dp^B} = \frac{1}{2}\left(1 \pm \frac{\sqrt{q}}{\sqrt{9q-16\alpha(1-\alpha+p^B)}}\right).$$

We obtain that $\frac{dp^A\left(p^B\right)}{dp^B} > 0$ in the third equality if and only if $q > 2(1-\alpha+p^B)$, that is, under pure vertical differentiation. Intuitively, in the case of horizontal dominance, the platform A reacts by decreasing it's price when p^B increases while under the case of vertical dominance, platform A reacts increasing their price.

The best reply functions of platform B correspond three polynomials of degree three relatively to p^A and of degree six relatively to α. Focusing on the polynomial that belongs to the real space $\mathbb{R}$, we obtain:

$$p^B\left(p^A\right) = \frac{3q-16\alpha(1+p^A)+\frac{-9q^2-288q\alpha^2-16\alpha^2(-1+p^A+3\alpha)^2}{\gamma}}{24\alpha}.$$

with $\gamma = \gamma(\alpha, q, p^A)$ representing a polynomial of degree three relatively to p^A and of degree six relatively to α. Because of this polynomial, we are not able to display closed-form solutions. Thus, we apply the theorem of Debreu, Glicksberg and Fan (1952) (see, for example, pp. 34 of Fudenberg and Tirole (1991) [12]) to mention that an equilibrium exists for such parameter region corresponding to the intermediate equilibrium. □

References

[1] Argenziano, R. (2008), "Differentiated networks: equilibrium and eficiency", *RAND Journal of Economics*, Vol. 39.3, pp. 747-769.

[2] Armstrong, M. (2006), "Platform Competition in Two-Sided Markets", *RAND Journal of Economics*, Vol. 37, pp. 668-691.

[3] Caillaud, B. and B. Jullien (2003), "Chicken & egg: Competition among intermediation service providers", *RAND Journal of Economics*, Vol. 34.2, pp. 309-328.

[4] Caplin, A. and B. Nalebuff (1991), "Aggregation and imperfect competition: On the existence of equilibrium", *Econometrica*, Vol. 59, pp. 25-59.

[5] D.Aspremont, C., J. J. Gabszewicz, and J. F. Thisse (1979), "On Hotelling's" Stability in competition", *Econometrica: Journal of the Econometric Society*, Vol. 47.5, pp. 1145-1150.

[6] Debreu, G. (1952), "A social equilibrium existence theorem", *Proceedings of the National Academy of Sciences of the United States of America*, Vol. 38.10, pp. 886-893.

[7] Di Comite, F., Thisse, J. F. and H. Vandenbussche (2011), "Verti-zontal differentiation in monopolistic competition", *Centro Studi Luca d'Agliano Development Studies Working Paper N^o. 322*.

[8] Economides, N. (1989), "Symmetric equilibrium existence and optimality in differentiated product markets.", *Journal of Economic Theory*, Vol. 47.1, pp. 178-194.

[9] Economides, N. (1993), "Quality variations in the circular model of variety differentiated products", *Regional Science and Urban Economics*, Vol. 23.2, pp. 235-257.

[10] Economides, N. (1996), "The economics of networks", *International Journal of Industrial Organization*, Vol. 14.6, pp.673-699.

[11] Economides, N. and J. Tåg (2012), "Network neutrality on the Internet: A two-sided market analysis", *Information Economics and Policy*, Vol. 24.2, pp. 91-104

[12] Fudenberg, D. and J. Tirole (1991), "Game Theory", Cambridge, Massachusetts, MIT Press.

[13] Fudenberg, D. and J. M. Villas-Boas (2006), "Behavior-Based Price Discrimination and Customer Recognition", *In: T. Hendershott (Ed.) Handbooks in Information Systems: Economics and Information Systems, Amsterdam, The Netherlands: Elsevier. Chapter 7.*

[14] Gabszewicz, J. J. and J. Resende (2012), "Differentiated credence goods and price competition", *Information Economics and Policy*, Vol. 24.3, pp. 277-287.

[15] Gabszewicz, J. J., and X. Y. Wauthy (2012), "Nesting horizontal and vertical differentiation", *Regional Science and Urban Economics*, Vol. 42, pp. 998-1002.

[16] Gabszewicz, J. J., and X. Y. Wauthy (2014), "Vertical product differentiation and two-sided markets", *Economics Letters*, Vol. 123.1 , pp. 58-61.

[17] Gabszewicz, J.J., Amir, R. and J. Resende (2013), "Thematic clubs and the supremacy of network externalities", *Journal of Public Economic Theory*, *Forthcoming*, doi: 10.1111/jpet.12081.

[18] Hotelling, H. (1929), "Stability in competition", *Economic Journal*, Vol. 39, pp. 41.57.

[19] Moorthy, K. S. (1988), "Product and price competition in a duopoly", *Marketing Science*, Vol 7.2, pp. 141-168.

[20] Motta, M. (2004), "Competition Policy. Theory and Practice", *Cambridge University Press.*

[21] Mussa, M. and S. Rosen (1978), "Monopoly and product quality", *Journal of Economic Theory*, Vol. 18.2, pp. 301-317.

[22] Neven, D. and J. F. Thisse (1990), "On quality and variety competition", *JJ Gabszewicz, J. F. Richard, L. Wolsey (Eds.), Economic Decision Making: Games, Econometrics, and Optimization. Contributions in the Honour of Jacques H. Drèze, North-Holland, Amsterdam*, pp. 175-199.

[23] Rochet, J. C. and J. Tirole (2003), "Platform competition in two-sided markets", *Journal of the European Economic Association*, Vol. 1.4, pp. 990-1029.

[24] Rochet, J. C. and J. Tirole (2006), "Two-sided markets: a progress report", *RAND Journal of Economics*, Vol. 37.3, pp. 645-667.

[25] Serfes, K. and E. Zacharias, (2012), "Location Decisions of Competing Networks", *Journal of Economics & Management Strategy*, Vol. 21.4, pp. 989-1005.

[26] Shaked, A. and J. Sutton (1982), "Relaxing price competition through product differentiation", *The Review of Economic Studies*, Vol. 49.1, pp. 3-13.