Documentos de Trabalho
Working Paper Series

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NIPE WP 13 / 2008
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URL:
http://www.eeg.uminho.pt/economia/nipe

* NIPE – Núcleo de Investigação em Políticas Económicas – is supported by the Portuguese Foundation for Science and Technology through the Programa Operacional Ciência, Tecnologia e Inovação (POCI 2010) of the Quadro Comunitário de Apoio III, which is financed by FEDER and Portuguese funds.
Price Discrimination with Private and Imperfect Information

Rosa Branca Esteves†

March 2008

Abstract

This paper investigates the competitive and welfare effects of information quality improvements in markets where firms can price discriminate after observing a private and noisy signal about a consumer’s brand preference. I show that firms charge more to customers they believe have a brand preference for them, and that this price has an inverted-U shaped relationship with the signal’s accuracy. In contrast, the price charged after a disloyal signal has been observed falls as the signal’s accuracy rises. While industry profit and welfare fall as price discrimination is based on increasingly more accurate information, the reverse happens to consumer surplus. The model is also extended to a public information setting. For any level of the signal’s accuracy, moving from public to private information, will boost industry profit and welfare and reduce consumer surplus.

1 Introduction

“Dynamic pricing is a new version of an old practice: price discrimination. It uses a potential buyer’s electronic fingerprint—his record of previous purchases, his address, maybe the other sites he has visited—to size up how likely he is to balk if the price is high. If the consumer looks price-sensitive, he gets a bargain; if he doesn’t he pays a premium.”

Paul Krugman (2000)

The last years have witnessed dramatic advances in information technologies that have enhanced the firms’ ability to gather, storage and use a vast amount of consumer-specific data. This in turn has led firms to become increasingly able to recognise customers and to tailor different prices to customers with different past purchasing profiles. While this price discrimination practice has been termed as behaviour-based price discrimination by the economics literature (e.g. Fudenberg and Tirole (2000)), in the Internet jargon it as been named dynamic pricing. In fact, in September 2000, the Washington Post, reported that the book retailer Amazon was conducting dynamic pricing as it was charging different DVDs prices to first-time visitors and to registered customers. Notwithstanding that the development of information technologies has

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*This paper is part of my PhD thesis at Oxford University. I am extremely grateful to Mark Armstrong (my supervisor) for helpful discussions and criticisms. Thanks for comments are also due to Paul Klemperer and Robin Mason. The financial support of the Portuguese Science Foundation and University of Minho is gratefully acknowledged. Any errors are my own.

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greatly improved the capability of firms for gathering and using consumer-specific information for price discrimination purposes, customer recognition may, in practice, be imperfect rather than perfect, either due to insufficient and/or imprecise data or inaccurate statistical inferences.

When firms rely on imperfect information to segment the market, they will recognise customers with a less than perfect probability (i.e., imperfectly), which means that some customers will be wrongly recognised. This in turn will lead firms to offer some consumers the wrong intended price. Therefore, the incapability of firms to perfectly predict whether a consumer belongs to one or other segment will affect the profit and welfare effects of price discrimination based on customer recognition. Furthermore, in practice, it is often the case that firms segment customers on the basis of private information (e.g. on the basis of internal sources of customer information such as firms’ transaction databases that record individual customers’ purchase histories), implying that they don’t know for sure how will the rivals classify a particular customer.\(^1\)

The issue at the heart of this paper is basically to investigate how profit, consumer surplus and welfare evolve as the firms’ ability to recognise customers—and to price discriminate accordingly—gradually improves. Thus, the paper hopes to shed some light on the following issues. Do firms benefit from price discrimination based on more accurate consumer-specific information? Does a firm have an incentive to exchange its private information with its competitors? Do consumers benefit when firms have access to more accurate information about them? Should government regulation act to restrict information collection and information sharing for price discrimination practices?

In order to provide an answer to the previous issues, the paper addresses a stylised model in a Hotelling fashion where two firms A and B sell their products directly to consumers whose brand loyalty towards the right-hand firm (firm A) is indexed by their location along an interval. Each firm’s private information comes from the observation of a signal that is not seen by the rival firm, which in this case informs whether a customer is loyal to brand A or loyal to brand B. To analyse the competitive effects of price discrimination based on imperfect information I assume that the signal may be more or less accurate.\(^2\)

Consider the following example. Two on-line booksellers, say Amazon (brand A) and Barnes&Noble (brand B), know that potential consumers have different degrees of brand loyalty. Although firms might not have the capacity to identify the extent of each individual consumer brand preference, they can be able through the lens of their own information to estimate (with a less than perfect probability) whether a given customer will prefer their brand or rather the rival’s one. Since each firm recognise customers on the basis of its private and imperfect signal, it is uncertain about the signal observed by the rival and thereby about the price the rival will offer to a particular customer. It may happen that both firms observe the same type of signal, for instance that the customer is loyal to Amazon. In this case Barnes&Noble would have an incentive to offer that customer a better deal in order to entice him to buy its product. Although at a first glance it seems intuitive that Amazon would have an incentive to extract more surplus from a customer recognised as loyal, the fact that both firms receive the same signal acts to intensify competition, thereby limiting Amazon’s incentive to charge a high price to a

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\(^1\)Firms can also rely on consumer data acquired from specialised information vendors, such as Double Click and I-Behavior. These firms collect and sell consumer-specific information that usually includes an individual’s purchasing history, interests, income and so on.

\(^2\)When we log on say on Amazon’s webpage its computers can use specific algorithms to predict (in general with some randomness) which consumers are likely to accept a high price and which are likely to reject it, and then the firm sets prices accordingly.
customer perceived as loyal. On the other hand, it may also happen that firms observe different
signals. If Amazon receives a signal that informs that the customer is loyal to Amazon, while
Barnes&Noble observes a signal that says the consumer prefers its brand, then the customer is
misrecognised by one of the firms and both firms charge a price tailored to a loyal customer.
This gives rise to a misrecognition effect which acts to soften price competition.\footnote{The model
addressed in this paper fits well pricing policies that will be possible through the mobile wireless
technologies. Particularly relevant will be the possibility of firms setting their prices according
to the geographic location of consumers—i.e., to set different prices to consumers with different
different location-based information. In general, a typical cell phone that is turned on sends out signals every ten minutes and identifies the location of the nearest cell tower. These signals can then be used to determine a cell phone user’s location so that even without the precise auto-location technologies (e.g. GPS), a user’s location can be predicted fairly precisely. Thus, after receiving a signal more or less accurate about each individual’s location, a firm might be able to price discriminate between consumers located near its store and consumers located closer to a rival’s store. To read more on mobile wireless technologies and consumer uses see for instance \url{www.ftc.gov/bcp/reports/wirelesssummary.pdf}}

In order to measure the effects of price discrimination on the basis of different information
structures, I first analyse the benchmark case where price discrimination is not permitted, either
because the signal has no informational content or because price discrimination is illegal (Section 3).
Subsequently, I look at the equilibrium behaviour of firms when price discrimination is based
on private and imperfect information (Section 4). When price discrimination is allowed firms
will always set higher prices to consumers recognised as loyal than to those consumers they
believe have a brand preference for the rival product (Proposition 2).

The effects of information improvements on equilibrium outcomes is analysed in Section 5.
Regarding prices, we will see that the price charged to customers recognised as loyal has an
inverted-U relationship with the signal’s accuracy. When the signal’s accuracy is relatively low,
a customer recognised as loyal will face a price above the non-discrimination level, the reverse
happens when information accuracy is high. In contrast, the price charged to a customer recog-
nised as disloyal is always below the non-discrimination level, and even lower the more accurate
is the signal (Corollary 1). Because more precise information gives rise to less uncertainty in
the marketplace and to a more aggressive behaviour, we will see that less accurate information
is good for industry profit (Corollary 3).

The welfare analysis is carried out in Section 6. There we will see that though consumers are
better off as price discrimination is based on more accurate information (Corollary 4), industry
profit and welfare are unambiguously worse off with improvements in information precision
(Proposition 3).

This paper will also investigate the effects of price discrimination with public information
(Section 8). In doing so, it sheds some light on whether it would be in the interest of firms to
exchange their private information with their rivals. Because firms will price more aggressivly
under public than under private information we will see that consumers are expected to pay
lower prices under public information (Corollary 5) which then leads to lower equilibrium profit
(Corollary 6). Therefore, while consumers benefit when firms have access to the same piece
of information, the reverse happens for firms. In other words, moving from private to public
information is good for consumers, but bad for industry profit and welfare (Proposition 6).

The remainder of the paper is organised as follows. I finish this Introduction with a brief
review of the more relevant literature. Then Section 2 lays out the formal model with private
information. Section 7 derives the Bayesian Nash equilibrium with public information. Concluding
remarks appear in section 9.
Related literature  The model addressed in this paper blends features from the literature on price discrimination in imperfectly competitive markets. Particularly, it is closely related to the recent vein of research on behaviour-based price discrimination.

We have seen that the main contribution of this paper is to add to the previous literature a stylised theoretical model that encompasses situations where firms have some uncertainty about consumer preferences as well as about the information the rival is using for segmenting the market. One way of dealing with this possibility is by introducing some randomness in the accuracy of a firm’s private information (i.e., firms do not observe the same piece of information). In doing so, I assume that based on their own records on consumers’ past behaviour each firm knows only with some probability whether a consumer “belongs to its turf” or to the rival’s. One important implication of this assumption is that markets are no longer completely separate; and therefore one interesting feature of the present model is that some consumers will be misrecognised and will receive the wrong intended price.

Thisse and Vives (1988) show that when firms have the same piece of information about each individual consumer (i.e. each firm observes a public fully accurate signal of each consumer’s brand preference), and firms base their prices on this observed signal (i.e. firms set personalised prices), then each consumer is a completely separate market to be contested. As a result, they show that price discrimination may intensify competition leading all prices and profits to fall compared to the case where price discrimination is not possible. Thus, it can be said that the model proposed in this paper extends the Thisse and Vives’s model in the sense that even though a firm cannot observe the true brand preference of each individual consumer, it is able to observe a noisy private signal of a consumer’s brand preference, and thereby to set its prices accordingly. Indeed, the model addressed here is a natural way of thinking of situations where firms set personalised prices.

Other models have extended the Thisse and Vives’s analysis to frameworks where, although firms are unable to observe the brand preference of individual consumers, they are able to recognise them only as their own customers or as the rival’s customers. These models have assumed that by observing the consumers’ past purchasing decisions, firms obtain a public and perfectly accurate signal of whether a consumer prefers its brand or the rival’s one. When price discrimination is allowed, each firm offers a different price to loyal and to the rival’s customers. However, because information is fully precise and public there is neither misrecognition of consumers nor uncertainty about the rival’s information.

In these group of models market segmentation may be due to the existence of exogenous switching costs (e.g. Chen (1997) and Taylor (2003)), or due to exogenous brand preferences (e.g. Villas-Boas (1999), Fudenberg and Tirole (2000) and Shaffer and Zhang (2000)). As said, these models assume that firms observe the same piece of information and thereby that consumers are classified in the right segment. For instance, in Fudenberg and Tirole (2000) after a firm has observed whether or not a consumer bought its product previously, it is able, in period 2, to segment the market into old (i.e. loyal) customers and rival’s (i.e. disloyal) customers. Once one firm recognises a customer as a loyal it must be the case that the competitor recognises that customer as a disloyal one. Hence, market segmentation is based on public information thereby implying that firms are not uncertain about the information the rival is using for setting its discriminatory prices. When it comes to evaluate the profitability of price discrimination

\footnote{See Stole (2003) and Armstrong (2006).}
\footnote{See Fudenberg and Villas-Boas (2006) for a comprehensive review}
\footnote{In Thisse and Vives all prices fall when consumer brand preferences are uniformly distributed.}
Fudenberg and Tirole also find that price discrimination leads to a prisoner’s dilemma. Because each firm tries to poach each other’s customers, price discrimination acts to intensify competition leading all segment prices to fall as well as profits.

Villas-Boas (1999) analyses an infinite-horizon model with overlapping generations of consumers, where although firms are able to recognise old customers, they are not able to distinguish a first-time customer in the market from one that bought the rival’s product previously. In this case, each firm has private information concerning its old customers. That is, firms know with certainty whether a customer is an old one or not, but when they face a new customer they are uncertain about whether that customer is a first-time customer to the market or a rival’s previous customer. Even in this context, Villas-Boas shows that each firm’s temptation to try to attract the rival’s previous customers makes both firms cut prices in relation to the situation where price discrimination cannot occur. Hence he also finds that price discrimination leads to lower equilibrium prices and profits for all competing firms. (This result was also obtained in models within the switching cost approach like Chen (1997) and Taylor (2003).) Because this paper abstracts from any previous competition leading consumers to have a brand preference from one of the firms, it could be view as the second-period game of say Fudenberg and Tirole’s model. Nevertheless it is different from the latter model as it assumes that each firm’s signal is not fully accurate and it is its private information.

Finally, the analysis has some connections with Chen, Narasimhan and Zhang’s (2001) study of individual marketing with imperfect targetability. (They define targetability as a firm’s ability to predict the preferences and purchase behaviours of individual consumers.) Specifically, they propose a duopoly model where there are three segments of consumers; some consumers are captive to one or the other firm and the remaining consumers are switchers. Additionally, they assume that each firm has information only about its own captive consumers and switchers, and that each firms identifies (with a less than perfect probability) a given customer as a captive or a switcher. In their model firms compete only for the switchers. When firms can separate consumers and engage in price discrimination they increase the price for captive consumers and reduce the price for switchers. However, when targetability is not perfect it may happen that a captive consumer receives the price tailored to a switcher and vice-versa. In this way, a low level of targetability tends to soften price competition in the marketplace as firms try to avoid that some captive consumers receive a very low price. For this reason they show that profits may increase with improvements in targetability when it departs from a low enough level. Nevertheless, when the firms’ targetability becomes higher, firms can separate almost perfectly captive and switchers and, therefore, the ensuing competition for the switchers outweighs the surplus extraction rents from captive consumers. As a result of that, as firms’ targetability is sufficiently high further improvements in targetability will intensify competition and lead to a prisoner’s dilemma as well.

2 The model

Suppose two firms, A and B, sell competing brands of a good produced, without loss of generality, at zero marginal cost. The total number of consumers in the market is normalised to one. A consumer wishes to buy a single unit either from firm A or B. The consumer’s valuation for the product, $v$, is sufficiently high so that nobody stays out of the market. Consumers are heterogeneous in brand loyalty, defined as the minimum difference between the prices of two
competing brands necessary to induce a consumer to buy his less preferred brand. A consumer’s brand loyalty towards brand A is represented by a parameter $l$ uniformly distributed on the interval $[-rac{1}{2}, rac{1}{2}]$, where $l \leq v$. In so doing, all else equal, a consumer with $0 < l \leq \frac{1}{2}$ prefers brand A, while a consumer with $-rac{1}{2} \leq l < 0$ prefers brand B. A type $l$ consumer who buys his preferred brand, say brand $i$, enjoys a net surplus equal to $v - p_i$; if he buys his less preferred brand, say $j$, his net surplus is $v - p_j - |l|$, where $i, j = A, B$.

Consumers have private information about their types and firms are not able to observe the degree of brand loyalty of individual customers. However, assume that each firm possesses consumer-specific imperfect private information—acquired either from the firm’s transaction databases that record the customers’ individual previous purchases or from specialised information vendors—that enables it to infer (with a less than certain probability) whether each customer favours it or its rival. In other words, suppose that for each consumer a firm observes a private signal (i.e. not seen by the rival), that informs whether that customer is loyal to brand A (signal $\alpha$) or loyal to brand B (signal $\beta$). This implies that after a signal has been observed for a given customer, each firm classifies imperfectly that customer as a loyal or a disloyal. Because each firm’s signal is not perfect, it is uncertain about the loyalty of each individual customer, and so customers are recognised imperfectly which means that market segmentation is also imperfect. A more accurate signal is associated with better information in the sense of Blackwell (1951), which is then reflected in a higher ability of firms to classify correctly potential customers. When price discrimination is permitted, a firm pricing strategy consists of choosing a price to a customer it believes is likely to prefer its good and choosing a different price to a customer perceived as one that favours the rival’s product.

In short, conditional on a given consumer true type, suppose that firm A and B observe an independent private noisy signal $s_i \in \{\alpha, \beta\}$ where $i = A, B$ about that consumer’s brand preference. While $\alpha$ informs that the consumer is prefers brand A, $\beta$ informs that the consumer favours brand B. Assume further that it is common knowledge that the probability of each signal conditional on each consumer’s brand loyalty $l$ is given by,

$$q(l) = \Pr (s_i = \alpha \mid l),$$

$$1 - q(l) = \Pr (s_i = \beta \mid l).$$

Finally, assume that $q(l)$ is increasing in $l$, that is the greater is the degree of brand loyalty of a particular consumer, the higher is the probability of a firm observing a loyal signal. As an example, when say firm A’s signal is based on data on consumer past purchasing behaviour, it is more likely that firm A observes a loyal signal for a consumer that bought its product many times in the past than for a customer that bought few times or did not buy at all from from A in the past. For the sake of simplicity consider that

$$q(l) = \frac{1}{2} + bl, \quad 0 \leq b \leq 1,$$

where $b$ measures the signal’s accuracy. The signal discloses no information when $b$ is equal to zero, in which case firms have no way to distinguish customers. In contrast, the signal discloses increasingly more accurate information as $b$ approaches 1, thereby allowing firms to better recognise customers. For intermediate values of $b$ some consumers are incorrectly classified by firms, i.e. some consumers loyal to brand A are misrecognised as loyal to brand B and vice-versa.
**Updating beliefs**  After observing a given signal for a particular customer firms update their own beliefs over that customer’s true type and form beliefs about the rival’s signal for that same customer. The density function of $l$, denoted by $f(l)$ (which is in this case equal to 1) and $q(l)$ are common knowledge of both firms and thus form the basis of their prior and posterior beliefs. Hence, after receiving signal $s_i \in \{\alpha, \beta\}$, using Bayes rule, each firm’s posterior belief about a customer’s brand loyalty degree is given by the conditional density function $h_{s_i}(l)$ where

$$h_{\alpha}(l) = \Pr(l \mid s_i = \alpha) = \frac{\Pr(s_i = \alpha \mid l) f(l)}{\Pr(s_i = \alpha)} = \frac{q(l) f(l)}{\Pr(s_i = \alpha)},$$

(4)

$$h_{\beta}(l) = \Pr(l \mid s_i = \beta) = \frac{\Pr(s_i = \beta \mid l) f(l)}{\Pr(s_i = \beta)} = \frac{[1 - q(l)] f(l)}{\Pr(s_i = \beta)},$$

(5)

and

$$\Pr(s_i = \alpha) = \int_{-\frac{1}{2}}^{\frac{1}{2}} q(l) f(l)dl = \frac{1}{2},$$

(6)

$$\Pr(s_i = \beta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 - q(l)] f(l)dl = \frac{1}{2}.$$  

(7)

Now, given signal $s_i = k$, firm $i$ believes that firm $j$ observed signal $s_j = \alpha$ with probability $\rho_k = \Pr(s_j = \alpha \mid s_i = k)$. Therefore,

$$\rho_{\alpha} = \Pr(s_j = \alpha \mid s_i = \alpha) = \frac{\Pr(s_j = \alpha, s_i = \alpha)}{\Pr(s_i = \alpha)} = 2\lambda_{\alpha\alpha}$$

(8)

$$\rho_{\beta} = \Pr(s_j = \alpha \mid s_i = \beta) = \frac{\Pr(s_j = \alpha, s_i = \beta)}{\Pr(s_i = \beta)} = 2\lambda_{\beta\alpha}$$

(9)

where

$$\lambda_{kr} = \Pr(s_i = k, s_j = r); \quad k, r = \{\alpha, \beta\}.$$  

Conditional on $l$ the signals observed by firms are independently distributed. It follows that

$$\Pr(s_i = k, s_j = r \mid l) = \Pr(s_i = k \mid l) \Pr(s_j = r \mid l)$$

and,

$$\lambda_{kr} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pr(s_i = k, s_j = r \mid l) f(l)dl.$$  

(10)

Therefore,

$$\lambda_{\alpha\alpha} = \Pr(s_i = \alpha, s_j = \alpha) = \int_{-\frac{1}{2}}^{\frac{1}{2}} [q(l)]^2 f(l)dl = \frac{1}{4} + \frac{b^2}{12}.$$  

(11)

$$\lambda_{\beta\beta} = \Pr(s_i = \beta, s_j = \beta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 - q(l)]^2 f(l)dl = \frac{1}{4} + \frac{b^2}{12}. $$

(12)

$$\lambda_{\alpha\beta} = \Pr(s_i = \alpha, s_j = \beta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} q(l) [1 - q(l)] f(l)dl = \frac{1}{4} - \frac{b^2}{12}. $$

(13)

Since both firms observe signal $\alpha$ with equal probability it follows that $\lambda_{\beta\alpha} = \lambda_{\alpha\beta}$.  

---

Notice that $h_{s_i}(l)$ satisfies the property $\int_{-\frac{1}{2}}^{\frac{1}{2}} h_{s_i}(l)dl = 1$. 

7
Lemma 1. For any level of the signal’s accuracy it follows that \( \rho_{\alpha} \geq \rho_{\beta} \).

Proof. See the Appendix.

In words, when firm A receives signal \( \alpha \) for a particular consumer it believes that it is more likely that the rival receives the same type of signal for that consumer. The more accurate is the signal the higher is the likelihood of say firm B has observed signal \( \alpha \) given that firm A has observed signal \( \alpha \). (Note that this probability is equal to \( \rho_{\alpha} = \frac{1}{2} + \frac{b^2}{2r} \).) Hence, more accurate signals reduce each firm’s uncertainty about the rival’s private information.

Firms also form beliefs about the loyalty degree of a given consumer after signals \( s_i \) and \( s_j \) have been observed. This is given by the conditional density function \( g_{kr}(l) \) where,\(^8\)

\[
g_{kr}(l) = \Pr (l \mid s_i = k, s_j = r) = \frac{\Pr (s_i = k \mid l) \Pr (s_i = r \mid l) f(l)}{\Pr (s_i = k, s_j = r)}.
\]

Thus,

\[
g_{a\alpha}(l) = \Pr (l \mid s_i = \alpha, s_j = \alpha) = \frac{[q(l)]^2 f(l)}{\lambda_{a\alpha}}, \tag{14}
\]

\[
g_{a\beta}(l) = g_{\beta\alpha}(l) = \Pr (l \mid s_i = \alpha, s_j = \beta) = \frac{q(l)(1 - q(l)) f(l)}{\lambda_{a\beta}}, \tag{15}
\]

\[
g_{\beta\beta}(l) = \Pr (l \mid s_i = \beta, s_j = \beta) = \frac{(1 - q(l))^2 f(l)}{\lambda_{\beta\beta}}. \tag{16}
\]

Finally,

\[
G_{kr}(x) = \Pr (l < x \mid s_i = k, s_j = r) = \int_{\frac{x}{2}}^{x} g_{kr}(l)dl. \tag{17}
\]

3 Benchmark case: non-discrimination

To better evaluate the competitive effects of price discrimination based on imperfect and private information about consumer brand preferences, consider first the benchmark case where price discrimination cannot occur either because it is illegal or because the signal has no informational content (i.e. \( b = 0 \)). Here the setup is analogous to a standard symmetric Hotelling model, playing the loyalty parameter \( l \) the same role as the transportation cost. If firms cannot price discriminate, in the symmetric equilibrium they will set the non-discrimination price \( p_N = \frac{1}{2} \).

With non-discrimination, equilibrium profit per firm is \( \pi_N = \frac{4}{9} \), consumer surplus is \( CS_N = v - \frac{1}{2} \), and total welfare is \( W_N = 2\pi_N + CS_N = v \).

4 Price discrimination with private information

Consider now the case where firms set a price to consumers they think are likely to prefer their product and a different price to those consumers they believe have a preference for the rival’s product. Specifically, when firm A sees signal \( \alpha \) for a particular customer, it recognises the consumer as a perceived loyal and will set price \( p_L \). In contrast, when firm A observes signal

\(^8\)Notice that \( g_{kr}(l) \) satisfies the property \( \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} g_{kr}(l)dl = 1 \).
\( \beta \), the consumer is believed to be a disloyal customer and firm A will set price \( p_D \). However, due to imperfect information some consumers will be misrecognised and will be offered the wrong intended price. Further, because each firm classifies customers on the basis of its private information, the way each firm segments the market is also its private information. Thus, each firm is uncertain about the price offered by its competitor to a particular customer.

Formally, upon observing signal \( s_i \in \{ \alpha, \beta \} \), firm A chooses \( p_{s_A}^A \in \{ p_{\alpha}^A, p_{\beta}^A \} \) and firm B chooses \( p_{s_B}^B \in \{ p_{\alpha}^B, p_{\beta}^B \} \). For the sake of simplicity, assume \( s_A = k \) and \( s_B = r \), where \( k, r = \{ \alpha, \beta \} \). I will use the Bayesian Nash equilibrium as the solution concept. Equilibrium prices are obtained by solving the ensuing maximisation problems:

\[
\max_{p_{s_A}^A} E [\pi^A | s_A = k] \tag{18}
\]

and

\[
\max_{p_{s_B}^B} E [\pi^B | s_B = r] \tag{19}
\]

where

\[
E (\pi^A | s_A = \alpha) = p_{\alpha}^A \sum_{r \in \{\alpha, \beta\}} \Pr (p_{\alpha}^A < p_{\beta}^B + l | s_A = \alpha, s_B = r) \Pr (s_B = r | s_A = \alpha) \nonumber
\]

\[
= p_{\alpha}^A \left( \rho_{\alpha} \left[ 1 - G_{\alpha\alpha} (p_{\alpha}^A - p_{\alpha}^B) \right] + (1 - \rho_{\alpha}) \left[ 1 - G_{\alpha\beta} (p_{\alpha}^A - p_{\beta}^B) \right] \right), \tag{20}
\]

and,

\[
E (\pi^A | s_A = \beta) = p_{\beta}^A \sum_{r \in \{\alpha, \beta\}} \Pr (p_{\beta}^A < p_{\beta}^B + l | s_A = \beta, s_B = r) \Pr (s_B = r | s_A = \beta) \nonumber
\]

\[
= p_{\beta}^A \left( \rho_{\beta} \left[ 1 - G_{\beta\alpha} (p_{\beta}^A - p_{\alpha}^B) \right] + (1 - \rho_{\beta}) \left[ 1 - G_{\beta\beta} (p_{\beta}^A - p_{\beta}^B) \right] \right). \tag{21}
\]

Symmetric expressions hold for firm B. Considering for instance the perspective of firm A, first-order conditions are:

\[
\frac{\partial E (\pi^A | s_A = \alpha)}{\partial p_{\alpha}^A} = \rho_{\alpha} \left[ 1 - G_{\alpha\alpha} (p_{\alpha}^A - p_{\alpha}^B) \right] + (1 - \rho_{\alpha}) \left[ 1 - G_{\alpha\beta} (p_{\alpha}^A - p_{\beta}^B) \right] - p_{\alpha}^A \left[ \rho_{\alpha\alpha} (p_{\alpha}^A - p_{\alpha}^B) + (1 - \rho_{\alpha}) g_{\alpha\beta} (p_{\alpha}^A - p_{\beta}^B) \right] = 0
\]

\[
\frac{\partial E (\pi^A | s_A = \beta)}{\partial p_{\beta}^A} = \rho_{\beta} \left[ 1 - G_{\beta\alpha} (p_{\beta}^A - p_{\alpha}^B) \right] + (1 - \rho_{\beta}) \left[ 1 - G_{\beta\beta} (p_{\beta}^A - p_{\beta}^B) \right] - p_{\beta}^A \left[ \rho_{\beta\alpha} (p_{\beta}^A - p_{\alpha}^B) + (1 - \rho_{\beta}) g_{\beta\alpha} (p_{\beta}^A - p_{\beta}^B) \right] = 0
\]

From both conditions it follows that:

\[
p_{\alpha}^A = \frac{\rho_{\alpha} \left[ 1 - G_{\alpha\alpha} (p_{\alpha}^A - p_{\alpha}^B) \right] + (1 - \rho_{\alpha}) \left[ 1 - G_{\alpha\beta} (p_{\alpha}^A - p_{\beta}^B) \right]}{\rho_{\alpha}\alpha + (1 - \rho_{\alpha}) g_{\alpha\beta} (p_{\alpha}^A - p_{\beta}^B)} \tag{22}
\]
and
\[ p^A_\beta = \frac{\rho_\beta \left[ 1 - G_{\beta_\alpha} \left( p^A_\delta - p^B_\alpha \right) \right] + (1 - \rho_\beta) \left[ 1 - G_{\beta_\beta} \left( p^A_\beta - p^B_\beta \right) \right]}{\rho_\beta g_{\beta_\alpha} \left( p^A_\delta - p^B_\alpha \right) + (1 - \rho_\beta) g_{\beta_\beta} \left( p^A_\beta - p^B_\beta \right)}. \] (23)

Similar expressions hold for firm B. Second-order conditions are also satisfied as can be seen in the proof of Proposition 1 in the Appendix provided.

**Competitive Equilibrium** Since the game is symmetric one must look at an equilibrium solution where \( p^A_\alpha = p^B_\beta \) and \( p^A_\beta = p^B_\alpha \). In doing so, the above system is reduced to two equations with two unknowns. Denoting as \( p_L \) the price quoted by firms to customers recognised as loyal and as \( p_D \) the price offered to customers recognised as disloyal it follows that \( p_L = p^A_\alpha = p^B_\beta \) and \( p_D = p^A_\beta = p^B_\alpha \). Unfortunately, since \( G \) is a cubic function the model does not allow for closed-form solutions. However, imposing the condition \( y = p_L - p_D \) gives the answer implicitly. We get the following result.

**Proposition 1.** When firms price discriminate on the basis of private and imperfect information about consumer band preferences, the symmetric Bayesian Nash equilibrium (BNE) in prices is given by
\[ p_L = \frac{\frac{1}{3} + \frac{1}{5}b - \frac{1}{5}y - \frac{1}{2}by^2 - \frac{1}{4}b^2y^3}{\frac{1}{2} + by + b^2y^2}, \] (24)

and
\[ p_D = \frac{\frac{1}{3} - \frac{1}{5}b + \frac{1}{5}y + \frac{1}{2}by^2 + \frac{1}{4}b^2y^3}{\frac{1}{2} + by + b^2y^2}. \] (25)

**Proof.** See the Appendix.

Notice that the symmetric interior solution presented in Proposition 1 is the only possible equilibrium solution. Despite the cubic equation in \( y \), it is easy to see that there is only one real root for all values of \( b \in [0, 1] \). Therefore, the symmetric equilibrium derived is unique. Further,

**Proposition 2.** (i) When the signal is non-informative (i.e. \( b = 0 \)), price discrimination is unfeasible and equilibrium prices are \( p_L = p_D = \frac{1}{2} \).

(ii) For any informative signal (i.e. \( b > 0 \)) it follows that \( p_L > p_D \).

(iii) The more informative is the signal the greater is the difference between the two discriminatory prices, i.e. \( p_L - p_D \) increases with \( b \).

Using (24) and (25) part (i) is easily obtained. To prove (ii) and (iii) Figure 1 plots \( y = p_L - p_D \) as an implicit function of \( b \),\(^9\) showing that for any level of the signal’s accuracy \( y > 0 \).

Proposition 2 claims that firms will charge more to customers they believe have a brand preference for their product than to customers that, from their perspective, are more price-sensitive, which are those customers recognised as disloyal. Furthermore, it predicts that the gap between the loyal and disloyal prices will be greater as the accuracy of information improves. When firms price discriminate on the basis of imperfect and private information they are uncertain not only about the true type of each individual consumer but also about the signal seen by the rival for the same consumer.

---

\(^9\)More precisely, \( y \) is defined implicitly as follows: \[ y + 2by^2 + \frac{2}{5}b^2y^3 - \frac{1}{4}b = 0. \]
Consider the case where information’s accuracy is low. On the one hand, it is more likely that a customer recognised as disloyal (loyal) turns out to be a true loyal (disloyal). In this case firms cannot clearly distinguish a loyal from a disloyal customer and thus they have few incentives to reduce the price to a customer generating a disloyal signal. On the other hand, it is less likely that both firms receive equal signals. When say firm A observes signal $\alpha$ there is a good chance that firm B observes signal $\beta$, thereby implying that both firms charge the loyal price. Thus, when the signal’s precision is low firms are more uncertain about each consumer’s type and the rival’s private information. This acts to soften price competition in the market leading to a smaller gap between loyal and disloyal prices.

In contrast, when information’s accuracy improves it is less likely that firms receive the wrong signal and it is more likely that both firms observe the same signal for a particular consumer.\textsuperscript{10} Consumers can be recognised more accurately by both firms which now compete more aggressively in prices. Thus, the gap between loyal and disloyal prices is higher.

5 Effects of information improvements

Advances in information technologies have allowed firms to gather, storage and use detailed consumer-specific databases for pricing and other marketing purposes (e.g. advertising). This section aims to shed some light on the following issues. What are the competitive effects of price discrimination with imperfect and private information? How do prices and profits evolve as the firms’ ability to recognise customers more accurately improves? Do firms benefit when they have access to more accurate information about consumer brand preferences? What about consumer welfare?

5.1 Prices

Using the equilibrium solutions presented in (24) and (25) it follows that:

\textsuperscript{10} Notice that the probability of a customer be in fact loyal say to firm A, after signal $\alpha$ has been observed, is given by $h_\alpha(1)$ (see equation (4)) which is increasing in $b$. Note also that the probability of both firms observing the same signal, i.e. $\lambda_{AB}$, given by equation (11), is increasing in $b$, meaning that as information’s accuracy increases it is also more likely that firms classify customers in the same way.
Corollary 1. When $b > 0$ then:

(i) the price charged to a customer recognised as disloyal is always below the non-discrimination price, i.e. $p_D < p_N$.

(ii) The price charged to a customer recognised as loyal is above the non-discrimination price when the signal’s accuracy is not too high, and below the non-discrimination price when the signal’s precision is high.

Figure 2: Relationship between prices and the signal’s accuracy

Figure 2 shows the behavior of equilibrium prices as information accuracy improves: $p_D$ (dashed line), $p_L$ (thin line) and $p_N$ (bold line). There we see that the price charged to customers recognised as loyal has an inverted-U shaped relationship with the signal’s accuracy. Specifically, we can see that each firm charges a consumer identified as loyal a price higher than its non-discrimination counterpart when the signal’s accuracy is (approximately) below 0.5 and after that point it charges that consumer a price lower than the non-discrimination level.\textsuperscript{11} On the contrary, a consumer recognised as disloyal always faces a price below the non-discrimination level, and this price falls as the private signal becomes more accurate.

In order to explain these findings it is important to have in mind that there are different effects at work when price discrimination is based on imperfect and private information. There is the “surplus extraction effect” through which price discrimination allows a firm to extract greater surplus from those consumers willing to pay more for its product (i.e. its loyal customers). In a competitive setting there is also the “business stealing effect” as the ability to set discriminatory prices gives firms an incentive to reduce the price to disloyal consumers as away to entice them. Apart from these two effects, the present analysis suggests that when price discrimination is based on inaccurate information there is another effect at work, namely the “misrecognition effect”.

When information is imperfect firms may classify consumers incorrectly thereby leading them to sometimes offer the wrong intended price. Consider the following example. Suppose that an individual consumer prefers brand A. Under imperfect and private information four events are

\textsuperscript{11}Numerical analysis shows that $p_L$ reaches its maximum value approximately at 0.5154 for $b \approx 0.25$ and is equal to the non-discrimination level for $b \approx 0.5$. When $b = 1$ the price charged to a perceived loyal customer falls to 0.44 (lower than the non-discrimination level).
relevant: (i) both firms observe signal \( \alpha \) and the consumer is correctly recognised by both firms; (ii) while firm A observes signal \( \alpha \), firm B observes signal \( \beta \) and the consumer is misrecognised by firm B; (iii) firm A observes signal \( \beta \) and firm B observes signal \( \alpha \), in which case the consumer is misrecognised by firm A, and (iv) both firms observe the wrong signal, that is signal \( \beta \), and the consumer is misrecognised by both firms. It is important to stress that with private information, misrecognition of consumers can occur either when both firms misrecognise (i.e. when both receive the wrong signal) or when only one of them misrecognises (i.e. when both receive different signals). We will see that, with public information, misrecognition of consumers will only occur when both firms observe the wrong signal.

Corollary 1 predicts that the price to a consumer recognised as disloyal is always below its non-discrimination counterpart and that this price decreases as information’s quality improves. The intuition for this finding is straightforward. When information precision is low, misrecognition of consumers is more likely and a firm has less incentives to decrease its price to a consumer that generates a disloyal signal, because there is a good chance that the consumer turns out to be a true loyal consumer. However, as information’s accuracy increases it becomes less likely that firms observe a wrong signal for a given consumer. In other words, better information means that it is more likely that a given consumer turns out to be a disloyal one after a disloyal signal has been observed. Hence, as the private signal’s precision rises the “misrecognition effect” becomes smaller leading the “business stealing effect” to become larger. In this way, the price to a consumer recognised as disloyal falls as information’s precision increases.

One of the most striking findings of this paper is that there is an inverted-U shaped relationship between the quality of information and the price to a consumer recognised as loyal. In order to explain this non-monotonic relationship we must have in mind that there have to be opposing forces at work, with a balance between the forces that changes as the parameter \( b \) increases.

Apart from strategic reasoning, when a firm receives a loyal signal for a given consumer it has an incentive to raise its price as a way of appropriating greater surplus from that consumer. When information is imperfect the expected surplus that a firm can extract after a loyal signal has been observed depends on the probability that a correct signal of a consumer type has been observed. Thus, the firm expects to extract greater surplus when the probability of classifying incorrectly a consumer decreases. Because misrecognition decreases as \( b \) increases the “surplus extraction effect” becomes larger as the quality of information improves.

Consider now the strategic interaction between firms. At low levels of the private signal’s accuracy the firm’s uncertainty about the true type of a particular consumer and the rival’s private information is high. Suppose that firm A receives signal \( \alpha \) for a particular consumer. As the signal’s precision rises firm A is more certain about the true type of that consumer and so it has more incentives to raise the price to a customer recognised as loyal. On the other hand, when the signal’s precision departs from a low level of precision, firm A acknowledges that even when firm B receives signal \( \alpha \) for that consumer it prices less aggressively because the consumer may turn out to be its own. However, as the quality of information becomes increasingly more precise, although firm A realises that it is more likely that it receives a correct signal, it also acknowledges that it is more likely that the rival firm receives the same signal and a correct signal as well. The “misrecognition effect” becomes smaller, firms compete more aggressively for each consumer leading the “business stealing effect” to become increasingly larger.

Summing up, for sufficiently low levels of the private signal’s accuracy (i.e. \( b \lesssim 0.25 \)) the misrecognition effect acts to soften price competition and reinforces the surplus extraction effect.
Since increases in $b$ increase the informational content of the signal and competition is not so intense, the price to a consumer recognised as loyal first increases with improvements in information’s accuracy. However, as information becomes increasingly more precise, despite the fact that firms are more likely to receive a correct signal it is also more likely that both firms receive the same signal (i.e. the “misrecognition effect” becomes increasingly smaller). As the “misrecognition effect” becomes weaker and weaker firms compete more aggressively and the “business stealing effect” becomes larger. This explains why the price to a customer recognised as loyal is a decreasing function of the quality of information for $b \gtrless 0.25$. Nevertheless, when $0.25 \lesssim b \lesssim 0.5$, the “misrecognition effect” allows the “surplus extraction effect” to dominate the “business stealing effect”. In contrast, when information becomes increasingly more and more accurate (i.e. $b \gtrsim 0.5$) the role of the “misrecognition effect” becomes so small that the “business stealing effect” dominates the “surplus extraction effect”, thus the price to a loyal signal falls below its non-discrimination counterpart.

As expected a very accurate signal intensifies price competition thereby explaining why both prices are below the non-discrimination level. This latter result is not new. We know from the previous literature that when each firm’s strong market is the rival’s weak market, price discrimination acts to intensify competition thereby lowering all segment prices (e.g. Thïsse and Vives (1988), Chen (1997), Villas-Boas (1999) and Fudenberg and Tirole (2000)). The novelty of our analysis is that it suggests that price discrimination which is based on relatively imprecise information will in general allow firms to appropriate a higher proportion of surplus from each consumer recognised as loyal. While intuition suggests that for a monopolist more precise information is always better as it increases its capability to extract more consumer surplus, in a duopolistic setting information relatively inaccurate may act to soften price competition, thus allowing firms to appropriate more surplus from loyal consumers.

5.2 Probability of winning a customer

When say Amazon infers that a particular customer is a loyal one it may be interested in determining what is the probability of winning that customer with a loyal tailored price, taken into account that say Barnes&Noble also offers that customer a price based on its beliefs about that customer’s brand preference. This section looks on how this probability evolves as information becomes more accurate. We first analyse the case where price discrimination is allowed.

**Price discrimination is allowed** Due to symmetry let $\gamma_L$ and $\gamma_D$ denote the probability of a firm winning a customer with a loyal and a disloyal signal. Using, for instance, the perspective of firm A,

$$
\gamma_L = \Pr(\text{firm A wins customer | } s_A = \alpha) = \sum_{r \in \{\alpha, \beta\}} \Pr(p^A_\alpha < p^B_r + l \mid s_A = \alpha, s_B = r) \Pr(s_B = r \mid s_A = \alpha) = [1 - G_{aa}(y)] \rho_\alpha + [1 - G_{a\beta}(0)] (1 - \rho_\alpha)
$$
and,

\[ \gamma_D = \Pr(\text{firm } A \text{ wins customer } | s_A = \beta) \]
\[ = \sum_{r \in \{\alpha, \beta\}} \Pr(p_A^P < p_B^P \mid s_A = \alpha, s_B = r) \Pr(s_B = r \mid s_A = \alpha) \]
\[ = [1 - G_{\beta\alpha}(0)] \rho_\beta + [1 - G_{\beta\beta}(-y)] (1 - \rho_\beta). \]

After some algebra one gets that,

\[ \gamma_L = \frac{1}{2} + \frac{1}{4} b - \frac{1}{2} y - by^2 - \frac{2}{3} b^2 y^2 = \frac{1}{2} + b^2 y^3 + by^2 + \frac{1}{2} y, \] \tag{26}

and

\[ \gamma_D = \frac{1}{2} - \frac{1}{4} b + \frac{1}{2} y + by^2 + \frac{2}{3} b^2 y^3 = \frac{1}{2} - b^2 y^3 - by^2 - \frac{1}{2} y. \] \tag{27}

**Corollary 2.** The greater is the accuracy of each firm’s private signal the greater is the probability of a firm winning a customer with a loyal price and the lower is the probability of a firm winning a customer with a disloyal price.

It is straightforward to verify that for any \( b > 0 \) it immediately follows \( y > 0 \) and \( \gamma_L > \gamma_D \). Further, while \( \gamma_L \) is strictly increasing in \( b \), \( \gamma_D \) is strictly decreasing in \( b \).\footnote{It is easy to check that as \( \frac{d \gamma_L}{db} > 0 \) then \( \frac{d \gamma_L}{dy} > 0 \) and \( \frac{d \gamma_D}{dy} < 0 \) for any \( b \in [0, 1] \).} As expected when the signal has non-informational content—i.e. when \( b = 0 \)—price discrimination is not feasible and so \( \gamma_L = \gamma_D = 0.5 \). As the quality of information improves, the greater is the likelihood of firms facing in fact a loyal customer after observing a loyal signal and, therefore, the higher is the probability of firms winning a customer with a loyal signal. The reverse happens for the probability of winning a customer with a disloyal price.

**Price discrimination is illegal** In this case \( p_A^\alpha = p_A^\beta, \ p_B^\alpha = p_B^\beta \) and as result \( y = 0 \). Again symmetry allows us to ease notation and to denote as \( \gamma^N_L \) and \( \gamma^N_D \) the probability of each firm winning a customer after observing a loyal and a disloyal signal, respectively, when discrimination is not allowed. Using (26) and (27) and the fact that \( y = 0 \) one gets that:

\[ \gamma^N_L = \frac{1}{2} + \frac{1}{4} b, \] \tag{28}

and

\[ \gamma^N_D = \frac{1}{2} - \frac{1}{4} b. \] \tag{29}

With non-discrimination, firms set the price \( \frac{1}{2} \) regardless of observing signal \( \alpha \) or \( \beta \). Therefore, the probability of winning a customer with a loyal signal only depends on \( b \) and due to our assumptions the greater is \( b \) the greater is that probability. The reverse happens for the probability of winning a consumer with a disloyal signal. Obviously the probability of a firm winning a customer with the non-discriminatory price is always equal to \( \frac{1}{2} \). From the comparison between equations (26) and (28) and using the fact \( y > 0 \), it is easy to see that \( \gamma^N_L > \gamma_L \) while \( \gamma^N_D < \gamma_D \). Under non-discrimination consumers have no incentive to swap brands, thus \( \gamma^N_L > \gamma_L \). Conversely, because each firm tries to attract those customers who prefer the rival’s product by offering them a lower price it is obvious that the probability of winning a customer after observing a disloyal signal should be higher than its no-discrimination counterpart.
5.3 Expected number of Inefficient Shoppers

When firms are allowed to price discriminate, some consumers might have an incentive to swap to their less preferred brand. The expected number of inefficient shoppers denoted \( EIS \) can be obtained as follows:\(^{13}\)

\[
EIS = (\gamma^N_L - \gamma_L) \Pr(s_A = \alpha) + \Pr(s_B = \beta)
\]

or,

\[
EIS = \frac{1}{2} y + by^2 + \frac{2}{3} b^2 y^3. \tag{30}
\]

The number of customers who buy inefficiently in equilibrium under price discrimination depends on the informativeness of the signal and on the difference between prices, that is \( y \). As expected, when discrimination is not allowed every consumer buys his most preferred brand, thereby implying that no consumer buys inefficiently. In contrast, when price discrimination is permitted it is easy to see that more accurate information gives rise to more inefficient shopping in equilibrium. (It is straightforward to see that for \( b > 0 \) it follows that \( \frac{dEIS}{db} > 0 \).) We have seen that as information’s precision improves the difference between \( p_L \) and \( p_D \) is higher. As a result of that, those customers with a smaller brand loyalty degree—i.e. those located in the middle—will have more incentives to buy the wrong brand. It is easy to see that when the signal reaches its maximum level of accuracy, i.e. when \( b = 1 \), approximately 12.5% of customers buy inefficiently, meaning that firms can attract some consumers that prefer the rival’s brand.

5.4 Profits

This section investigates how profits respond to information’s accuracy improvements. Let \( E\pi_L \) and \( E\pi_D \) represent, respectively, each firm expected profit from a loyal and a disloyal signal.

\[
E\pi_L = p_L \left( \frac{1}{2} + \frac{1}{2} b - \frac{1}{2} y - by^2 - \frac{2}{3} b^2 y^3 \right), \tag{31}
\]

\[
E\pi_D = p_D \left( \frac{1}{2} - \frac{1}{4} b + \frac{1}{2} y + by^2 + \frac{2}{3} b^2 y^3 \right). \tag{32}
\]

Given that \( \Pr(s_i = \alpha) = \Pr(s_i = \beta) = \frac{1}{2} \), each firm expected aggregate profit, denoted by \( E\Pi \) is equal to

\[
E\Pi = \frac{1}{2} (E\pi_L + E\pi_D). \tag{33}
\]

Corollary 3. (i) The profit from a loyal signal exhibits an inverted-U shaped relationship with the private signal’s accuracy. Conversely, the profit from a disloyal signal decreases as the signal’s accuracy improves.

(ii) Expected profit with private information is always lower than the non-discrimination profit and falls monotonically as the accuracy of the private signal rises.

\(^{13}\)Because the model abstracts from any previous competition, it is more convenient to adopt the term inefficient shoppers rather than the usual “switchers” which is more indicate to situations in which consumers choose one brand in one period and a different one in the other period.
Figure 3 illustrates the firm’s profit conditional on a loyal and a disloyal signal and the precision of the private signal. Obviously, the relationship between the profit from customers recognised as loyal or from consumers recognised as disloyal and the accuracy of the private information is strongly related to the relationship between the price conditional on each signal and the signal’s accuracy. In fact, we have seen before that when a firm’s signal is not very accurate—i.e. \( b \) approximately below 0.5—a firm is able to extract more surplus from customers recognised as loyal which clearly is good for profits. However, when the signal becomes increasingly more accurate firms compete more aggressively which leads to lower prices and profits. Similarly, because the price charged to customers perceived as disloyal always falls as the accuracy of information improves, the same happens to profits from perceived disloyal customers.

Look next at expected aggregate profit as firms rely on more accurate information. Figure 4 shows the expected aggregate profit with private information (thin line) and without discrimination (bold line). The extant literature has shown that in markets exhibiting best-response asymmetry, price discrimination might lead to all-out competition (e.g. Thisse and Vives (1988), Chen (1997) and Fudenberg and Tirole (2000)). However, this conclusion was obtained in the extreme case where firms observe the same piece of information and know for sure whether a customer is a loyal or a disloyal one. That is, this result was obtained in models where firms price discriminate on the basis of a perfect segmentation of the market (either on an individual basis, e.g. Thisse and Vives (1988); or on a group basis, e.g. Fudenberg and Tirole (2000)). The analysis developed so far adds to the existing literature new insights by analysing a market where competing firms are neither able to perfectly infer each customer’s type neither to know the rival’s private information for pricing purposes.
Figure 4: Expected aggregated profit

With some probability each firm’s strong perceived market may coincide, thereby soften price competition. (This is more likely to occur when $b$ is smaller.) However, Figure 4 shows that even in this case price discrimination is bad for profits. Although firms can charge higher prices to consumers recognised as loyal when the signal is not too precise, it also happens that the probability of winning a customer with this price is smaller. Thus, the expected surplus extraction benefit is not enough to overcome the reduction in profits that result from a lower price being charged to a consumer recognised as disloyal. Because more accurate information turns out into more aggressive pricing, the model shows firm and industry profit benefit when price discrimination is based on highly inaccurate information.

It is further worth stressing that our findings are also different from Chen, Narasimhan and Zhang (2001). In their model there are only three types of consumers, each firm has a captive group of consumers and firms only compete for switchers. Thus, they find that both firms might benefit from price discrimination based on more accurate information (which they designate as targetability) when the level of information precision is low. Specifically, they show that when firms depart from low levels of targetability, profits will increase as targetability improves. The intuition for their result is as follows. When targetability is not too high, improvements in it allow firms to extract more surplus from their captive customers as they are increasingly able to identify them. At the same time, because targetability remains at a relatively low level, firms can not fully separate switchers from captive customers which clearly softens price competition for the switchers. Thus, while in their model competing firms may benefit from more accurate information, here this is never the case.

As in the previous literature, in the present model when firms try to attract each other’s customers, the prisoner’s dilemma result always ensues. Firms would like to collectively commit to uniform pricing but individually each firm prefers to price discriminate. However, in the present model the information technology available for each firm acts as a restriction to more aggressive price discrimination, which of course is good for profits. Notice that when the information technology is fully uninformative there is a credible kind of commitment to uniform pricing, which benefits all competing firms. Thus, the current analysis and earlier work put forward that competing firms could all benefit from regulatory policies protecting consumer privacy, which would limit the firms’ ability to recognise customers and thereby to set discriminatory prices in
a more aggressive competitive context.

6 Welfare analysis

This section investigates the social welfare impacts of information improvements when price discrimination is allowed. In order to evaluate the effects of improvements in the signal’s accuracy on welfare, we carry out numerical simulations assuming, without any loss of generality, that \( v = 2 \). As usual expected social welfare is defined as the sum of expected industry profit and expected consumer surplus (ECS). Hence,

\[
EW = 2EI + ECS.
\] (34)

**Lemma 2.** Expected consumer surplus is given by:

\[
ECS = v - p_L\gamma_L - p_D\gamma_D - \frac{1}{4}y^2 - \frac{2}{3}by^3 - \frac{1}{2}b^2y^4.
\] (35)

**Proof.** See the Appendix.

As expected, when \( b = 0 \), expected consumer surplus is equal to consumer surplus with non-discrimination, that is \( v - p_N \).

**Corollary 4. (Consumer welfare)** (i) As a whole consumer surplus with discrimination is above the non-discrimination level, and it increases monotonically as the accuracy of the private signal increases.

(ii) Each individual consumer is increasingly better off as price discrimination is based on more accurate information.

Figure 5 shows expected consumer surplus as the signal’s accuracy improves. Thus, it confirms part (i) of corollary 4. Likewise Figure 6 confirms part (ii). This figure shows expected consumer surplus for each loyalty degree and for a fixed level of accuracy. Particularly the figure is plotted for \( b = 0.1 \) (lower bold line); \( b = 0.25 \) (thin line) and \( b = 1 \) (upper bold line).

**Figure 5: Expected consumer surplus**

![Figure 5](image)
When discrimination is not allowed (or when \( b = 0 \)) expected consumer surplus per consumer is constant, regardless each consumer loyalty degree \( l \). Particularly, when \( v = 2 \) expected consumer surplus per individual consumer equals 1.5. Although ECS remains approximately at the same level for tiny increases in \( b \), when the signal becomes increasingly informative we observe that ECS increases with \( b \). We have seen that as the signal becomes more informative price competition on a segment basis is more intense and thereupon prices fall. As a result, consumers can buy at better deals.

Despite consumers as a whole are unequivocally better off with price discrimination, we have seen that for not too accurate signals (\( b \lesssim 0.5 \)) some consumers are expected to face a higher price than that under non-discrimination. At a first glance this seems to suggest that not all consumers would benefit when price discrimination is based on low quality information. Nevertheless, Figure 6 above shows that even in the range where loyal customers are expected to pay a price above that under non-discrimination, expected consumer surplus is above its non-discrimination counterpart. Notice that \( p_L \) reaches its maximum value when \( b \) is approximately equal to 0.25 and even in this case expected consumer surplus for the most loyal customers \( l = \{-\frac{1}{2}, \frac{1}{2} \} \) is higher than ECS under non-discrimination. The intuition is as follows. In spite of for small levels of the signal's accuracy loyal customers are expected to pay higher prices, it is also true that with some positive probability they will be misrecognised (i.e. they will be classified as a disloyal consumer) and therefore they will have a chance to buy the product at the lowest price \( p_D \). Although a firm tends to charge higher prices to perceived loyal customers when \( b \) is low, it is also true that it is more likely that the firm mistakenly recognises a true loyal customer as a disloyal when \( b \) is low. As a result, expected consumer surplus is always above the non-discrimination level.

Look now at the expected social welfare. Using (33) and (35) it ensues that:

\[
EW = p_L \gamma_L + p_D \gamma_D + \left( v - p_L \gamma_L - p_D \gamma_D - \frac{1}{4} y^2 - \frac{2}{3} by^3 - \frac{1}{2} b^3 y^4 \right) 
\]

(36)

\[
= v - \left( \frac{1}{4} y^2 + \frac{2}{3} by^3 + \frac{1}{2} b^2 y^4 \right).
\]

Thus, we may establish the following proposition.
**Proposition 3.** Price discrimination is always good for consumers although bad for profits and overall welfare. Further, consumers are increasingly better off, while firms and welfare are increasingly worse off as price discrimination is based on more accurate private signals.

Equation (36) shows that welfare is equal to \( v \) minus the disutility incurred by those consumers who buy inefficiently. As \( y \) increases monotonically as the private signal rises, then, the higher is \( b \), the higher is \( \left( \frac{1}{4}y^2 + \frac{2}{3}by^3 + \frac{1}{2}b^2y^4 \right) \) and so the lower is welfare. Because with non-discrimination aggregate welfare equals \( v \), it is clear-cut that with discrimination expected welfare is always below the non-discrimination level. As firms become increasingly able to recognize customers, and to segment them more accurately (i.e., less imperfectly), the stronger is the damage of price discrimination on welfare. In fact, welfare reaches its minimum value when the signal’s accuracy reaches its maximum level. The reason is that the expected number of consumers that buy inefficiently in equilibrium reaches its maximum value when \( b = 1 \). Because in the present model there is no role for price discrimination to increase aggregate output, variations in welfare are uniquely explained by the costs incurred by those consumers that buy the wrong brand. Thus, improvements in the information required for price discrimination leads to more inefficient shopping which clearly is not good for welfare.\(^{14}\)

In short, as information accuracy improves profits and welfare move in the same direction but consumer surplus moves in an opposite direction. This suggests that any advice to a regulatory authority should take into account whether the target is welfare or solely consumer surplus. Restrictions protecting consumer privacy and limiting the efficacy of price discrimination would benefit industry profits and social welfare but at the expense of consumer welfare.

### 7 Price discrimination with public information

The analysis developed so far has assumed that firms price discriminate on the basis of private information. One interesting issue is to investigate how profits, consumer surplus and welfare evolve as we depart from a setting where price discrimination is based on private information to a setting where price discrimination is based on public information—i.e., a framework where both firms have the same piece of information. Looking at this issue will also shed light on whether a firm benefits or not from exchanging its private information with its rival for price discrimination purposes.

With that in mind this section extends the model to the case where both firms observe a public signal about each consumer’s brand preference. The signal is “public” in the sense that its actual realization is common knowledge to both firms. This means that although firm A observes \( s_A \) and firm B observes \( s_B \) each firm knows the signal of each other. Alternatively, this public information case could also fit a situation where both firms observe two public signals about a particular consumer’s brand loyalty rather than one. It can fit as well the case where firms exchange its customer information with its competitor.

To ease notation we denote by \( s \) the public signal observed by firms where \( s = (s_A = k, s_B = r) \) and \((k, r) = \{ (\alpha, \alpha), (\beta, \beta), (\alpha, \beta), (\beta, \alpha) \} \). Again, suppose that \( \alpha \) informs that the consumer is

\(^{14}\)Notice that this result is in contrast with that achieved in Thissee and Vives (1988). The reason is that in their model price discrimination based on perfect information gives rise to no welfare loss since no consumer actually switches in equilibrium. Here because firms can only segment consumers into loyal or disloyal (as in Fudenberg and Tirole (2000)) there are always some consumers who buy inefficiently. Therefore, here price discrimination based on better information increases the welfare loss.
loyal to firm A while $\beta$ informs that the consumer is loyal to firm B. In this scenario firms may observe a biased signal in favour of one firm, that is two signals informing that the consumer is either loyal to firm A or loyal to firm B (i.e. $(\alpha, \alpha)$ or $(\beta, \beta)$); or, alternatively, firms may observe a non-biased signal meaning that one of the signals reveals that the consumer prefers A, while the other reveals that the consumer prefers B (i.e. $(\alpha, \beta)$ or $(\beta, \alpha)$). Since now signals are public firms only need to update their beliefs about each consumer’s brand loyalty degree after observing signal $s$. Thus, using our previous computations $g_{rk} = \Pr(l \mid s = (r, k))$ is now the density function of $l$ conditional on the public signal observed by firms.

### 7.1 Competitive equilibrium

In the public information framework, after the public signal has been observed, each firm may classify a particular consumer into three different segments. The consumer can be recognised as (i) a strong loyal customer, (ii) a strong disloyal customer and as (iii) a non-biased consumer. Under price discrimination each firm tailors a different price for each different type of signal. Thus, if price discrimination is permitted, firm $i$ chooses simultaneously $p_i^s \in \{p_{\alpha\alpha}^i, p_{\alpha\beta}^i, p_{\beta\alpha}^i, p_{\beta\beta}^i\}$, $i = A, B$. These prices are the solution to the following problem

$$\max_{p_i^s \geq 0} E \left[ \pi^i \mid s \right], \quad (37)$$

where, for instance for firm A, one gets

$$E \left[ \pi^A \mid s = (\alpha, \alpha) \right] = p_{\alpha\alpha}^A \Pr \left( p_{\alpha\alpha}^A < p_{\alpha\alpha}^B + l \mid s = (\alpha, \alpha) \right) = p_{\alpha\alpha}^A \left[ 1 - G_{\alpha\alpha} (p_{\alpha\alpha}^A - p_{\alpha\alpha}^B) \right],$$

$$E \left[ \pi^A \mid s = (\beta, \beta) \right] = p_{\beta\beta}^A \Pr \left( p_{\beta\beta}^A < p_{\beta\beta}^B + l \mid s = (\beta, \beta) \right) = p_{\beta\beta}^A \left[ 1 - G_{\beta\beta} (p_{\beta\beta}^A - p_{\beta\beta}^B) \right],$$

$$E \left[ \pi^A \mid s = (\alpha, \beta) \right] = p_{\alpha\beta}^A \Pr \left( p_{\alpha\beta}^A < p_{\alpha\beta}^B + l \mid s = (\alpha, \beta) \right) = p_{\alpha\beta}^A \left[ 1 - G_{\alpha\beta} (p_{\alpha\beta}^A - p_{\alpha\beta}^B) \right],$$

$$E \left[ \pi^A \mid s = (\beta, \alpha) \right] = p_{\beta\alpha}^A \Pr \left( p_{\beta\alpha}^A < p_{\beta\alpha}^B + l \mid s = (\beta, \alpha) \right) = p_{\beta\alpha}^A \left[ 1 - G_{\beta\alpha} (p_{\beta\alpha}^A - p_{\beta\alpha}^B) \right].$$

Symmetric expressions hold for firm B. Due to symmetry, in a Bayesian Nash equilibrium it follows that $p_{\alpha\alpha}^A = p_{\beta\beta}^B$, $p_{\beta\alpha}^A = p_{\alpha\beta}^B$ and $p_{\alpha\beta}^A = p_{\beta\alpha}^B = p_{\alpha\alpha}^B = p_{\beta\beta}^A$. Following the same reasoning as before let $p_{LB}^{pub}$ and $p_{DB}^{pub}$ be, respectively, the equilibrium prices firms charge after observing a double loyal and a double disloyal signal, in the public information context. Likewise let $p_{NB}^{pub}$ be the price firms charge after observing a non-biased signal. Again, $p_{LB}^{pub} = p_{\alpha\alpha} = p_{\beta\beta}$ and $p_{DB}^{pub} = p_{\beta\beta}^B = p_{\alpha\alpha}^B$. After some algebra it is possible to establish the following propositions.
Proposition 4. When information is public, the Bayesian Nash equilibrium in prices is given by

\[ p_{L}^{\text{pub}} = \frac{1 + b - \frac{1}{2}b^2}{2} \left( \frac{1}{2} + b \frac{z}{z} \right)^2, \quad (38) \]

\[ p_{D}^{\text{pub}} = \frac{1 - b + \frac{1}{2}b^2 + \frac{1}{2}b^2}{2} \left( \frac{1}{2} + b \frac{z}{z} \right)^2, \quad (39) \]

where \( z = p_{L}^{\text{pub}} - p_{D}^{\text{pub}} \); and

\[ p_{NB}^{\text{pub}} = \frac{1}{2} - \frac{1}{6}b^2. \quad (40) \]

Proof. See the Appendix.

Proposition 5. (i) When the public signal is fully has no informational content (i.e. \( b = 0 \)), \( p_{L}^{\text{pub}} = p_{D}^{\text{pub}} = p_{NB}^{\text{pub}} = \frac{1}{2} \).

(ii) For any informative public signal, the price charged to customers perceived as loyal is always higher than the price charged to customers perceived as disloyal. That is, \( p_{L}^{\text{pub}} > p_{D}^{\text{pub}} \).

(iii) As the public signal becomes increasingly more precise, the higher is the difference between \( p_{L}^{\text{pub}} \) and \( p_{D}^{\text{pub}} \).

(iv) The more accurate is the public signal, the lower is the price charged to non-biased consumers.

(v) For any level of the public signal’s informativeness, \( p_{L}^{\text{pub}} > p_{NB}^{\text{pub}} > p_{D}^{\text{pub}} \).

Figure 7: Behaviour of equilibrium prices under public information.

Proof. Part (i) can be easily proved using (38), (39) and (40) and making \( b = 0 \). Part (iv) is also clear looking at (40). Part (iii) is proved by plotting \( z = p_{L}^{\text{pub}} - p_{D}^{\text{pub}} \) as an implicit function of \( b \), where \( b \in [0, 1] \). This is done in Figure 8. There the reader can observe that \( z \) is positive and monotonically increases with \( b \). Finally, Figure 7 proves part (v) by plotting \( p_{L}^{\text{pub}} \) (bold line), \( p_{NB}^{\text{pub}} \) (thin line) and \( p_{D}^{\text{pub}} \) (dots line). Intuition suggests that when firms are
more certain about the loyalty of a particular consumer they have more incentives to charge that consumer a higher price. The same reasoning applies when firms observe a double disloyal signal. Obviously when firms observe a non-biased signal it is equally likely that the consumer can turn out to be a truly loyal or disloyal. So, the price charged to a consumer with that type of signal should be below $p_{L}^{pab}$ but above $p_{D}^{pab}$.

Under public information, expected equilibrium profits conditional on each signal are equal to:

$$E_{\pi_L}^{pab} = \frac{1}{\lambda_{\alpha\alpha}^*} (p_{L}^{pab}) \left( \frac{1}{8} + \frac{1}{8} b + \frac{1}{24} b^2 - \frac{1}{4} z - \frac{1}{2} b z^2 - \frac{1}{3} b^2 z^3 \right), \quad (41)$$

$$E_{\pi_D}^{pab} = \frac{1}{\lambda_{\beta\beta}^*} (p_{D}^{pab}) \left( \frac{1}{8} - \frac{1}{8} b + \frac{1}{24} b^2 + \frac{1}{4} z + \frac{1}{2} b z^2 + \frac{1}{3} b^2 z^3 \right), \quad (42)$$

$$E_{\pi_{NB}}^{pab} = \frac{2}{\lambda_{\alpha\beta}^*} (p_{NB}^{pab}) \left( \frac{1}{8} - \frac{1}{24} b^2 \right) = p_{NB}^{pab} \left( \frac{1}{4} - \frac{1}{12} b^2 \right). \quad (43)$$

Due to symmetry each firm’s expected aggregated profit under public information denoted $E\Pi^{pab}$ equals:

$$E\Pi^{pab} = E_{\pi_L}^{pab} \Pr(\text{double loyal}) + E_{\pi_D}^{pab} \Pr(\text{double disloyal}) + E_{\pi_{NB}}^{pab} \Pr(\text{non-biased}).$$

8 Private versus public information: a comparative analysis

This section aims to examine whether firms, consumers and welfare benefit from price discrimination based on public rather than on private information. In this regard, in what follows we compare equilibrium prices, profits, consumer surplus and welfare attained with private and public information.

8.1 Equilibrium prices

Figure 8: Comparison between $y$ and $z$ in both information settings
To compare equilibrium prices upon observing a loyal and a disloyal signal, under both information settings, Figure 8 plots the difference between the highest and the lowest price under private information—namely, \( y \) (bold line)—and under public information—namely, \( z \) (dashed line). If price dispersion is measured by the range of prices, it can be said that the level of price dispersion is greater with public than with private information. Further, Figure 8 suggests that the level of price dispersion is greater as firms rely on more accurate information about consumer brand preferences.

Figure 9 illustrates the relationship between the price charged to customers recognised as loyal under private information (bold line) and under public information (dashed line).

Figure 9: Price to loyal customers

As before, it is interesting to look at the relationship between \( p_{L}^{pub} \) and the accuracy of information. As in the private information setting we observe that there is a non-monotonic relationship between the quality of information and the price to a loyal consumer in the public information framework. Note that with public and imperfect information, misrecognition of consumers only occurs when both firms observe the wrong signal. When information is very imprecise, an increase in the signal’s accuracy has the following effects on \( p_{L}^{pub} \). Suppose firm A observes a double \( a \) signal. On the one hand, when \( b \) increases, firm A is able to better recognise a loyal consumer and to extract more consumer surplus. On the other hand, firm B observes the same signal and tries to entice that consumer by offering him a disloyal price. When the level of accuracy raises but remains relatively low, although the signal becomes more informative it remains noisy so it is still highly likely that a firm mistakenly recognises a true loyal as a disloyal consumer. Due to this misrecognition effect a firm has less incentives to quote a low price to a consumer recognised as disloyal. This acts to soften price competition, allowing the price charged to a consumer recognised as loyal to increase initially with information improvements. However, as the information becomes more and more accurate firms have less doubts about each consumer’s type and therefore they behave more aggressively.

For sufficiently low levels of information’s precision, increases in \( b \) reduce the probability of firms receiving a wrong signal thereby allowing the “surplus extraction effect” to become stronger. Put differently, as the probability of misrecognition decreases the “surplus extraction effect” increases, allowing firms to charge more to consumers recognised as loyal. (This explains why the price to a loyal consumer increases as the quality of information improves when \( b \lesssim 0.2 \).)
However, as information becomes more and more accurate, firms are more likely to receive a
correct signal, the role of the “misrecognition effect” is increasingly smaller leading firms to com-
pete more aggressively in prices. The “business stealing effect” becomes larger explaining why
the price to a customer recognised as loyal is a decreasing function of the quality of information
when $b \geq 0.2$. In this way, as the signal is increasingly more precise (i.e. when $b \geq 0.4$) the
“business stealing effect” dominates the “surplus extraction effect” and in consequence the price
to a loyal consumer becomes smaller than its non-discrimination counterpart.

Figure 10 illustrates the relationship between the price to customers recognised as disloyal
under private information (bold line) and under public information (dashed line).

**Figure 10: Price to disloyal customers**

Numerical analysis allows us to establish the following corollary.

**Corollary 5.**  
(i) Moving from private to public information reduces the price to customers perceived as disloyal.

(ii) When the signal’s accuracy is not too low (i.e. $b \geq 0.2$) the price to a customer perceived as loyal is higher under private than under public information.

This result claims that for sufficiently accurate signals (i.e. $b \geq 0.2$) the prices to loyal and
disloyal consumers are lower under public than under private information. Although the current
analysis offers no clear cut results with respect to the comparison between the price to a loyal
consumer in both information regimes when the signal’s accuracy is too low, it shows that as
information’s accuracy becomes increasingly higher consumers perceived as loyal are expected to
pay higher prices under private than under public information. Numerical analysis shows that
while under private information the price to a loyal consumer reaches the no-discrimination level
for $b \cong 0.5$ under public information this happens for $b \cong 0.4$. Similarly, we find that whilst under
private information the price to a loyal consumer reaches its maximum value approximately at
0.5154 for $b \cong 0.25$ under public information the price to a loyal consumer reaches its maximum
value approximately at 0.5138 for $b \cong 0.2$.

The intuition is as follows. We have seen that in the private information setting firms are
uncertain about each consumer’s true type as well as about the rival’s private information.
The latter type of uncertainty disappears in the public information framework. Whilst under
private information a given consumer may be recognised differently by both firms, under public information this is no longer the case because firms always classify consumers in the same way. As a consequence, at least for sufficiently accurate signals, we expect firms to compete more aggressively in the public information game which then leads the price to customers recognised as loyal as well as the price to customers recognised as disloyal to become smaller than their private information counterparts.

8.2 Expected profits

Since firms will play more aggressively when they have access to the same piece of information about consumers the next result ensues.

**Corollary 6.** *Firms are better off when price discrimination is based on private information rather than on public information.*

Figure 11 shows how aggregate expected profit behaves as the signal becomes more precise under private information (bold line) and under public information (dashed line). As each firm profit with non-discrimination is equal to 0.25, it follows that price discrimination is bad for profits in both information regimes. Further, expected profit with public information is always below its private information counterpart. A common finding in the existing literature is that in markets exhibiting best-response asymmetry price discrimination with public and perfect information leads in general to a prisoner’s dilemma (e.g. Thisse and Vives (1988), Chen (1997), Fudenberg and Tirole (2000)). The current model predicts that being information public firms are better off when they rely on more imperfect information about consumer brand preferences.

**Figure 11: Expected aggregated profit**

![Figure 11: Expected aggregated profit](image)

**Corollary 7.** *(Information sharing)* *Under symmetry a firm would have no incentive to exchange its private consumer-specific information with its competitor.*

Intuitively, the model suggests that if firms exchange their information with their competitors for price discrimination purposes they will be worse off. In this way, any regulatory policy
restricting firms to disclose private information about their customers to rival firms would benefit firms’ profitability at the expense of consumer welfare.

8.3 Expected number of inefficient shoppers

**Lemma 3** The expected number of inefficient shoppers under public information denoted $EIS^{pub}$ is:

$$EIS^{pub} = \left( \gamma_{N}^{N,pub} - \gamma_{L}^{L,pub} \right) = \frac{1}{2}z + bz^2 + \frac{2}{3}b^2z^3. \quad (44)$$

**Proof.** See the Appendix.

**Corollary 8.** The number of customers that buy the wrong brand is higher under public than under private information, and it is increasingly higher as information becomes more accurate.

**Proof.** See the Appendix.

The intuition for this result is as follows. On the one hand, because price discrimination leads to more intense competition and to lower prices to consumers recognised as disloyal in the public information case, it is obvious that those consumers that are not extremely loyal to one of the brands (i.e. those located in the middle) will have more incentives to buy the wrong brand under public information. On the other hand, the groups of consumers that generate inefficiency in both information frameworks are those who generate two identical signals as in this case they will receive a different price from the two firms. (Notice that consumers who generate different signals will always buy efficiently under public and under private information because they will receive the same price from the two firms.) So it is clear that the level of inefficiency is greater in the public than in the private information game.

8.4 Overall welfare

**Lemma 4.** Expected consumer surplus with public information denoted $ECS^{pub}$ equals:

$$ECS^{pub} = v - p_{NB}^{p^b} - p_{L}^{N,pub} \gamma_{L}^{N,pub} - p_{D}^{L,pub} \gamma_{D}^{L,pub} - \frac{1}{4}z^2 - \frac{2}{3}bz^3 - \frac{1}{2}b^2z^4.$$ 

**Proof.** See the Appendix.

**Lemma 5.** Expected welfare with public information denoted $EW^{pub}$ is equal to:

$$EW^{pub} = v - \frac{1}{4}z^2 - \frac{2}{3}bz^3 - \frac{1}{2}b^2z^4. \quad (45)$$

**Proof.** See the Appendix.

**Proposition 6.** For any level of the signal’s accuracy, moving from private to public information has the following effects: (i) consumer surplus increases; (ii) profits fall, and (iii) welfare falls.
Proof. See the Appendix for the proof of (iii). Part (i) is proved graphically.

Figure 12 shows expected consumer surplus with private (bold line) and public information (dashed line). Again the figure is plotted for the case where \( v = 2 \).

![Figure 12: Expected consumer surplus](image)

To summarise, allowing firms to have access to public information or, equivalently, to exchange consumer-specific information for price discrimination purposes, intensifies price competition in the marketplace which is good for consumers although bad for profits and overall welfare. Regarding consumers, they are able to buy products at lower prices when firms have very accurate information about them and, moreover, when all firms have the same piece of information. As far as firms is concerned, they do not benefit from being able to recognize customers more perfectly and, further, they become even worse off when the rival observes the same piece of information. This suggests that when firms have access to private information about customers, they should guard it and not share their information with their competitors.

Thus, the present analysis is useful because it offers some criterion to assess how welfare evolves as price discrimination is based on more precise information as well as on private rather than on public information. Paradoxically, it predicts that welfare is unambiguously greater when price discrimination is based on private and imperfect information. Obviously, we need to be extremely careful in drawing any public policy on the basis of this result. If aggregate output increased with either more accurate information or when moving from private to public information, then the effects of such changes would need to be taken into account.

9 Conclusions

This paper has tried to provide a more complete picture of the prices, profits and welfare effects of price discrimination as information technologies gradually improve firms’ ability to recognise different types of consumers. The main contribution was to extend the previous literature by allowing firms to price discriminate on the basis of imperfect and private information. We have seen that while imperfect information tends to lead firms to misrecognise customers, private information gives rise to some uncertainty about the rival’s information for price discrimination.
Besides a customer may be wrongly recognised by some firm, he can also be recognised in a different way by the two firms. In this way, it was shown that the economic implications of price discrimination are in fact affected by the misrecognition of customers as well as by the inability of firms to perfectly predict how the rival is segmenting customers.

In the private information setting, it was shown that customers recognised as loyal pay always a higher price than customers recognised as disloyal. This result is in consonance with Amazon’s practice of charging more to old than to first-time customers. It was also shown that customers recognised as loyal are expected to pay prices above the non-discrimination level when information is not too accurate while the reverse happens when information becomes more precise. By contrast, it was found that customers recognised as disloyal are expected to pay lower prices under price discrimination, and increasingly lower prices the more accurate is each firm’s private signal.

In light of the above it was shown that price discrimination is good for consumers, but bad for profits and welfare. Moreover, it was shown that the better is the quality of each firm’s private information, the greater is the consumer surplus and the lower are profits and overall welfare. The model has put forward that collectively it is not in the interest of firms to improve the quality of their information about customers, even though individually a firm might prefer to do so. Remarkably, any public policy protecting consumer privacy, by restricting customer recognition, would benefit all competing firms at the expense of consumer welfare.

This paper has also analysed price discrimination under public and imperfect information. Extending the model in this direction has proved to be helpful to understand whether or not firms might have an incentive to share their private information with their rivals. It was shown that as a whole consumers are expected to benefit more with public than with private information. In contrast, industry profit falls when moving from private to public information. Because more consumers are expected to buy the “wrong” brand with public information, it was shown that welfare falls when moving from private to public information.

Notwithstanding the model addressed in this paper is far from covering all complex aspects of real markets, it has tried to offer a closer approximation of reality where the quality of consumer-specific information that firms have been using to implement their pricing strategies is increasingly improving thanks to advances in information technologies. albeit the results derived in this paper should be interpreted carefully, the analysis is worth pursuing as it provides clear intuitions about the profit, consumer surplus and welfare effects of price discrimination based on different information structures.

Appendix

This appendix collects the proofs and computations that were omitted from the text.

Proof of Lemma 1. If \( \rho_a \geq \rho_b \) one must observe that:

\[
2\lambda_{aa} \geq 2\lambda_{bb} \text{ or } \frac{1}{4} + \frac{b^2}{12} \geq \frac{1}{4} - \frac{b^2}{12}
\]

which is true \( \forall b \in [0, 1] \). Q.E.D.
Proof of Proposition 1. From the first-order conditions for both firms we obtain:

\[
p_A^* = \frac{\rho_a \left[ 1 - G_{aa} \left( p_A^* - p_B^* \right) \right] + (1 - \rho_a) \left[ 1 - G_{ab} \left( p_A^* - p_B^* \right) \right]}{\rho_a g_{aa} \left( p_A^* - p_B^* \right) + (1 - \rho_a) g_{ab} \left( p_A^* - p_B^* \right)}, \tag{46}
\]

\[
p_B^* = \frac{\rho_B \left[ 1 - G_{bb} \left( p_B^* - p_A^* \right) \right] + (1 - \rho_B) \left[ 1 - G_{bb} \left( p_B^* - p_A^* \right) \right]}{\rho_B g_{bb} \left( p_B^* - p_A^* \right) + (1 - \rho_B) g_{bb} \left( p_B^* - p_A^* \right)}, \tag{47}
\]

\[
p_A = \frac{\rho_a G_{aa} \left( p_A - p_B^* \right) + (1 - \rho_a) G_{ab} \left( p_A - p_B^* \right)}{\rho_a g_{aa} \left( p_A - p_B^* \right) + (1 - \rho_a) g_{ab} \left( p_A - p_B^* \right)}, \tag{48}
\]

and

\[
p_B = \frac{\rho_B G_{bb} \left( p_B - p_A^* \right) + (1 - \rho_B) G_{bb} \left( p_B - p_A^* \right)}{\rho_B g_{bb} \left( p_B - p_A^* \right) + (1 - \rho_B) g_{bb} \left( p_B - p_A^* \right)}. \tag{49}
\]

Considering for instance the perspective of firm A, second-order partial derivatives with respect to both prices are

\[
\frac{\partial^2 E \left( \pi^A \mid s_A = \alpha \right)}{\partial p_A^2} = -2\rho_a g_{aa} \left( p_A^* - p_B^* \right) - 2 \left( 1 - \rho_a \right) g_{ab} \left( p_A^* - p_B^* \right) - p_A^* \rho_a g_{aa} \left( p_A - p_B^* \right) - \rho_a \left( 1 - \rho_a \right) g_{ab} \left( p_A - p_B^* \right),
\]

\[
\frac{\partial^2 E \left( \pi^A \mid s_A = \beta \right)}{\partial p_B^2} = -2\rho_B g_{bb} \left( p_B^* - p_A^* \right) - 2 \left( 1 - \rho_B \right) g_{bb} \left( p_B^* - p_A^* \right) - p_B^* \rho_B g_{bb} \left( p_B - p_A^* \right) - \rho_B \left( 1 - \rho_B \right) g_{bb} \left( p_B - p_A^* \right)
\]

and,

\[
\frac{\partial^2 E \left( \pi^A \mid s_A = \alpha \right)}{\partial p_A \partial p_B} = 0.
\]

Symmetric conditions hold for firm B. Due to symmetry we are looking for an equilibrium where \( p_A^* = p_B^* = p_L \) and \( p_B^* = p_B^* = p_D \). Thus, making \( y = p_L - p_D \), and using without loss of generality the perspective of firm A, one gets:

\[
p_L = p_A^* = \frac{\rho_a \left[ 1 - G_{aa} \left( y \right) \right] + (1 - \rho_a) \left[ 1 - G_{ab} \left( 0 \right) \right]}{\rho_a g_{aa} \left( y \right) + (1 - \rho_a) g_{ab} \left( 0 \right)},
\]

\[
p_D = p_B^* = \frac{\rho_B \left[ 1 - G_{bb} \left( y \right) \right] + (1 - \rho_B) \left[ 1 - G_{bb} \left( -y \right) \right]}{\rho_B g_{bb} \left( y \right) + (1 - \rho_B) g_{bb} \left( -y \right)}.
\]

Then second-order partial derivatives for firm A are then equal to:

\[
\frac{\partial^2 E \left( \pi^A \mid s_A = \alpha \right)}{\partial p_A^2} = -2\rho_a g_{aa} \left( y \right) - 2 \left( 1 - \rho_a \right) g_{ab} \left( 0 \right) - p_L \rho_a g_{aa} \left( y \right) - p_L \left( 1 - \rho_a \right) g_{ab} \left( 0 \right) - p_L \rho_a g_{aa} \left( y \right) - p_L \left( 1 - \rho_a \right) g_{ab} \left( 0 \right) - p_L \rho_a g_{aa} \left( y \right) - p_L \left( 1 - \rho_a \right) g_{ab} \left( 0 \right)
\]

\[
= -2\rho_a g_{aa} \left( y \right) - 2 \left( 1 - \rho_a \right) g_{ab} \left( 0 \right) - p_L \rho_a g_{aa} \left( y \right) - p_L \left( 1 - \rho_a \right) g_{ab} \left( 0 \right).
\]

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and

\[
\frac{\partial^2 E (\pi^A | s_A = \beta)}{\partial \rho_{\beta}^2} = -2 \rho_{\beta} g_{\beta\alpha} (0) - 2 (1 - \rho_{\beta}) g_{\beta\beta} (-y) - p_D \rho_{\beta} g'_{\beta\alpha} (0) - p_D (1 - \rho_{\beta}) g'_{\beta\beta} (-y).
\]

Since,

\[
1 - G_{\alpha\alpha} (y) = \frac{1}{\lambda_{\alpha\alpha}} \int_y^{\frac{1}{2}} \left( \frac{1}{2} + bl \right)^2 dl
= \frac{1}{\lambda_{\alpha\alpha}} \left( \frac{1}{8} + \frac{1}{8} b + \frac{1}{24} b^2 - \frac{1}{4} y - \frac{1}{2} by^2 - \frac{1}{3} b^2 y^3 \right),
\]

\[
1 - G_{\alpha\beta} (0) = \frac{1}{\lambda_{\alpha\beta}} \int_0^{\frac{1}{2}} \left( \frac{1}{2} + bl \right) \left( \frac{1}{2} - bl \right) dl
= \frac{1}{\lambda_{\alpha\beta}} \left( \frac{1}{8} - \frac{1}{24} b^2 \right),
\]

\[
g_{\alpha\alpha} (y) = \frac{1}{\lambda_{\alpha\alpha}} \left( \frac{1}{2} + by \right)^2,
\]

\[
g_{\alpha\beta} (0) = \frac{1}{\lambda_{\alpha\beta}} \left( \frac{1}{4} \right),
\]

\[
1 - G_{\beta\beta} (-y) = \frac{1}{\lambda_{\beta\beta}} \int_{-y}^{\frac{1}{2}} \left( \frac{1}{2} - bl \right)^2 dl
= \frac{1}{\lambda_{\beta\beta}} \left( \frac{1}{8} - \frac{1}{8} b + \frac{1}{24} b^2 + \frac{1}{4} y + \frac{1}{2} by^2 + \frac{1}{3} b^2 y^3 \right),
\]

\[
1 - G_{\beta\alpha} (0) = \frac{1}{\lambda_{\beta\alpha}} \int_0^{\frac{1}{4}} \left( \frac{1}{4} - b^2 t^2 \right) dl
= \frac{1}{\lambda_{\beta\alpha}} \left( \frac{1}{8} - \frac{1}{24} b^2 \right),
\]

\[
g_{\beta\alpha} (0) = \frac{1}{\lambda_{\beta\alpha}} \left( \frac{1}{4} \right),
\]

and

\[
g_{\beta\beta} (-y) = \frac{1}{\lambda_{\beta\beta}} \left( \frac{1}{2} + by \right)^2.
\]

It is now easy to check that second-order conditions for a maximum are satisfied. Using the expressions derived above and the fact that

\[
g'_{\alpha\alpha} (y) = \frac{2b}{\lambda_{\alpha\alpha}} \left( \frac{1}{2} + by \right)
\]

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\[ g'_{\alpha\beta} (0) = g'_{\beta\alpha} (0) = 0 \]
\[ g'_{\beta\beta} (-y) = -\frac{2b}{\lambda_{\beta\beta}} \left( \frac{1}{2} + by \right) \]

it is straightforward to observe that:
\[
\frac{\partial^2 E \left( \pi^A \mid s_A = \alpha \right)}{\partial \rho_{\alpha}^{A^2}} = -2\rho_{\alpha} g_{\alpha\alpha} (y) - 2 \left( 1 - \rho_{\alpha} \right) g_{\alpha\beta} (0) - p_L p_{\alpha} g'_{\alpha\alpha} (y) \\
\quad = -4\lambda_{\alpha\alpha} \frac{1}{\lambda_{\alpha\alpha}} \left( \frac{1}{2} + by \right)^2 - 4 \left( \lambda_{\alpha\beta} \right) \frac{1}{\lambda_{\alpha\beta}} \left( \frac{1}{4} \right) - p_L 2\lambda_{\alpha\alpha} \frac{2b}{\lambda_{\alpha\alpha}} \left( \frac{1}{2} + by \right) \\
\quad = -4 \left( \frac{1}{2} + by \right)^2 - 1 - 4b p_L \left( \frac{1}{2} + by \right) < 0.
\]

and
\[
\frac{\partial^2 E \left( \pi^A \mid s_A = \beta \right)}{\partial \rho_{\beta}^{A^2}} = -2\rho_{\beta} g_{\beta\alpha} (0) - 2 \left( 1 - \rho_{\beta} \right) g_{\beta\beta} (-y) - p_D \left( 1 - \rho_{\beta} \right) g'_{\beta\beta} (-y) \\
\quad = -4\lambda_{\beta\alpha} \frac{1}{\lambda_{\beta\alpha}} \left( \frac{1}{4} \right) - 4\lambda_{\beta\beta} \frac{1}{\lambda_{\beta\beta}} \left( \frac{1}{2} + by \right)^2 - p_D \left( 2\lambda_{\beta\beta} \right) \left( -\frac{2b}{\lambda_{\beta\beta}} \left( \frac{1}{2} + by \right) \right) \\
\quad = -1 - 4 \left( \frac{1}{2} + by \right)^2 + 4b p_D \left( \frac{1}{2} + by \right) \\
\quad = -1 - 4 \left( \frac{1}{2} + by \right) \left( \frac{1}{2} + b \left( y - p_D \right) \right) .
\]

Despite the fact that \( b \left( y - p_D \right) < 0 \) we find that for \( 0 \leq b \leq 1, 0.41 \leq \frac{1}{2} + b \left( y - p_D \right) \leq 0.5 \) thus \( \frac{\partial^2 E \left( \pi^A \mid s_A = \beta \right)}{\partial \rho_{\beta}^{A^2}} < 0 \). In sum, since \( \frac{\partial^2 E \left( \pi^A \mid s_A = \alpha \right)}{\partial \rho_{\alpha}^{A^2}} < 0 \) and \( \left( \frac{\partial^2 E \left( \pi^A \mid s_A = \alpha \right)}{\partial \rho_{\alpha}^{A^2}} \right) \left( \frac{\partial^2 E \left( \pi^A \mid s_A = \beta \right)}{\partial \rho_{\beta}^{A^2}} \right) - \left( \frac{\partial^2 E \left( \pi^A \mid s_A = \alpha \right)}{\partial \rho_{\alpha}^{A^2} \partial \rho_{\beta}^{A^2}} \right)^2 > 0 \) it follows that second-order conditions for a maximum are as well satisfied.

After some algebra, we find that the prices firms set after observing a loyal and a disloyal signal are, respectively, as follows:

\[
p_L = \frac{\frac{1}{4} + \frac{1}{6} b - \frac{1}{3} y - \frac{1}{6} by^2 - \frac{1}{3} b^2 y^3}{\frac{1}{2} + by + b^2 y^2} \quad (50)
\]
\[
p_D = \frac{\frac{1}{4} - \frac{1}{6} b + \frac{1}{3} y + \frac{1}{6} by^2 + \frac{1}{3} b^2 y^3}{\frac{1}{2} + by + b^2 y^2} . \quad (51)
\]

Q.E.D.

**Proof of Lemma 2.** For each individual consumer with brand loyalty parameter \( l \in [- \frac{1}{2}, \frac{1}{2}] \) each firm observes a binary signal. Representing by \( (s_A = i, s_B = j) \) the signal observed by the two firms for a particular consumer, where \( i, j \in \{ \alpha, \beta \} \), we may analyse four possible scenarios with respective equilibrium prices:
\[
\begin{array}{|c|c|c|}
\hline
(s_A = i, s_B = j) & (p_i^A, p_j^B) & \Pr(s_A = i, s_B = j \mid l) \\
\hline
(s_A = \alpha, s_B = \alpha) & (p_L, p_D) & \Pr(s_A = \alpha, s_B = \alpha \mid l) = (q(l))^2 \\
(s_A = \alpha, s_B = \beta) & (p_L, p_L) & \Pr(s_A = \alpha, s_B = \beta \mid l) = q(l) (1 - q(l)) \\
(s_A = \beta, s_B = \alpha) & (p_D, p_D) & \Pr(s_A = \beta, s_B = \alpha \mid l) = (1 - q(l)) q(l) \\
(s_A = \beta, s_B = \beta) & (p_D, p_L) & \Pr(s_A = \beta, s_B = \beta \mid l) = (1 - q(l))^2 \\
\hline
\end{array}
\]

Therefore,

\[
ECS = (v - p_L) \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - q(l)) q(l) f(l) dl + (v - p_D) \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - q(l))^2 f(l) dl
\]

\[
+ \int_{0}^{\frac{1}{2}} \max \{v - p_L, v - p_D - l\} (q(l))^2 f(l) dl
\]

\[
+ (v - p_D) \int_{-\frac{1}{2}}^{0} (q(l))^2 f(l) dl
\]

\[
+ \int_{-\frac{1}{2}}^{\frac{1}{2}} \max \{v - p_L, v - p_D + l\} (1 - q(l))^2 f(l) dl
\]

\[
+ (v - p_D) \int_{0}^{\frac{1}{2}} (1 - q(l))^2 f(l) dl.
\]

After some algebra and using the fact that \( y = p_L - p_D \), one finds that

\[
ECS = (v - p_L) \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{1}{2} - bl \right) \left( \frac{1}{2} + bl \right) dl + \int_{y}^{\frac{1}{2}} \left( \frac{1}{2} + bl \right)^2 dl + \int_{-\frac{1}{2}}^{-y} \left( \frac{1}{2} - bl \right)^2 dl \right)
\]

\[
+ (v - p_D) \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{1}{2} - bl \right) \left( \frac{1}{2} + bl \right) dl + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{1}{2} + bl \right)^2 dl + \int_{-\frac{1}{2}}^{-y} \left( \frac{1}{2} - bl \right)^2 dl \right)
\]

\[
- \int_{0}^{y} \left( \frac{1}{2} + bl \right)^2 dl + \int_{-y}^{0} \left( \frac{1}{2} - bl \right)^2 dl,
\]

from which we get

\[
ECS = v - p_L \left( \frac{1}{2} + \frac{1}{4} b - \frac{1}{2} y - by^2 - \frac{2}{3} b^2 y^3 \right) - p_D \left( \frac{1}{2} - \frac{1}{4} b + \frac{1}{2} y + by^2 + \frac{2}{3} b^2 y^3 \right)
\]

\[- \frac{1}{4} y^2 - \frac{2}{3} by^3 - \frac{1}{2} b^2 y^4.
\]

Q.E.D.

**Proof of Proposition 4.** Given that firms may observe four different types of signals, upon observing signal \( s = (i, j) \), each firm chooses simultaneously a different price for the signal observed. Hence, firm \( i \) chooses \( p_i^j \in \left\{ p_{\alpha \alpha}^i, p_{\beta \beta}^i, p_{\alpha \beta}^i, p_{\beta \alpha}^i \right\} \), \( i = A, B \). These prices are the solution to the following problem

\[
\max_{p_i \geq 0} E \left[ \pi^i \mid s \right],
\]

(52)

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Considering for example, the perspective of firm A, first-order conditions for firm A are:

\[ \frac{\partial E(\pi^A | s = (\alpha, \alpha))}{\partial p_{\alpha\alpha}^A} = [1 - G_{\alpha\alpha} (p_{\alpha\alpha}^A - p_{\alpha\alpha}^B)] - (p_{\alpha\alpha}^A) g_{\alpha\alpha} (p_{\alpha\alpha}^A - p_{\alpha\alpha}^B) = 0, \]

\[ \frac{\partial E(\pi^A | s = (\beta, \beta))}{\partial p_{\beta\beta}^A} = [1 - G_{\beta\beta} (p_{\beta\beta}^A - p_{\beta\beta}^B)] - (p_{\beta\beta}^A) g_{\beta\beta} (p_{\beta\beta}^A - p_{\beta\beta}^B) = 0, \]

\[ \frac{\partial E(\pi^A | s = (\alpha, \beta))}{\partial p_{\alpha\beta}^A} = [1 - G_{\alpha\beta} (p_{\alpha\beta}^A - p_{\alpha\beta}^B)] - (p_{\alpha\beta}^A) g_{\alpha\beta} (p_{\alpha\beta}^A - p_{\alpha\beta}^B) = 0, \]

\[ \frac{\partial E(\pi^A | s = (\beta, \alpha))}{\partial p_{\beta\alpha}^A} = [1 - G_{\beta\alpha} (p_{\beta\alpha}^A - p_{\beta\alpha}^B)] - (p_{\beta\alpha}^A) g_{\beta\alpha} (p_{\beta\alpha}^A - p_{\beta\alpha}^B) = 0. \]

Second-order conditions are as well satisfied. Due to symmetry we are looking for an equilibrium where \( p_{\alpha\alpha}^A = p_{\beta\beta}^B, \ p_{\beta\beta}^A = p_{\alpha\alpha}^B \) and \( p_{\alpha\beta}^A = p_{\beta\alpha}^B = p_{\beta\alpha}^A = p_{\alpha\beta}^B = p_{\beta\alpha}^A = p_{\beta\alpha}^B \). Making \( z = p_{\alpha\alpha}^A - p_{\alpha\alpha}^B \) we have

\[ [1 - G_{\alpha\alpha} (z)] - (p_{\alpha\alpha}^A) g_{\alpha\alpha} (z) = 0 \]

from which it follows:

\[ p_{\alpha\alpha}^A = \frac{1 - G_{\alpha\alpha} (z)}{g_{\alpha\alpha} (z)}. \] (53)

We also have

\[ [1 - G_{\beta\beta} (-z)] - (p_{\beta\beta}^A) g_{\beta\beta} (-z) = 0 \]

from which we obtain

\[ p_{\beta\beta}^A = \frac{1 - G_{\beta\beta} (-z)}{g_{\beta\beta} (-z)}. \] (54)

Finally, from

\[ [1 - G_{\alpha\beta} (0)] - (p_{\alpha\beta}^A) g_{\alpha\beta} (0) = 0 \]

we get

\[ p_{\alpha\beta}^A = p_{\beta\alpha}^A = \frac{1 - G_{\alpha\beta} (0)}{g_{\alpha\beta} (0)}. \] (55)

As before,

\[ 1 - G_{\alpha\alpha} (z) = \frac{1}{\lambda_{\alpha\alpha}} \left( \frac{1}{8} + \frac{1}{8} b + \frac{1}{24} b^2 - \frac{1}{4} z - \frac{1}{2} b z^2 - \frac{1}{3} b^2 z^3 \right), \]

\[ 1 - G_{\alpha\beta} (0) = \frac{1}{\lambda_{\alpha\beta}} \left( \frac{1}{8} - \frac{1}{24} b^2 \right), \]

\[ g_{\alpha\alpha} (z) = \frac{1}{\lambda_{\alpha\alpha}} \left( \frac{1}{2} + b z \right)^2, \]

\[ g_{\alpha\beta} (0) = \frac{1}{\lambda_{\alpha\beta}} \left( \frac{1}{7} \right), \]

\[ 1 - G_{\beta\beta} (-z) = \frac{1}{\lambda_{\beta\beta}} \left( \frac{1}{8} - \frac{1}{8} b + \frac{1}{24} b^2 + \frac{1}{4} z - \frac{1}{2} b z^2 + \frac{1}{3} b^2 z^3 \right), \]
\[1 - G_{\beta \alpha} (0) = \frac{1}{\lambda_{\beta \alpha}} \left( \frac{1}{8} - \frac{1}{24} b^2 \right),\]
\[g_{\beta \alpha} (0) = \frac{1}{\lambda_{\beta \alpha}} \left( \frac{1}{4} \right),\]

and
\[g_{\beta \beta} (-z) = \frac{1}{\lambda_{\beta \beta}} \left( \left( \frac{1}{2} + b z \right)^2 \right),\]

Rearranging and using the fact \(p_{L}^{\text{pub}} = p_{A}^{\text{pub}} = p_{B}^{\text{pub}}\) and \(p_{D}^{\text{pub}} = p_{A}^{\text{pub}} = p_{B}^{\text{pub}}\) we find that:
\[p_{L}^{\text{pub}} = \frac{\frac{1}{5} + \frac{1}{8} b + \frac{1}{24} b^2 - \frac{1}{4} z - \frac{1}{2} b z^2 - \frac{1}{3} b^2 z^3}{\left( \frac{1}{2} + b z \right)^2},\]  
\[p_{D}^{\text{pub}} = \frac{\frac{1}{5} - \frac{1}{8} b + \frac{1}{24} b^2 + \frac{1}{4} z + \frac{1}{2} b z^2 + \frac{1}{3} b^2 z^3}{\left( \frac{1}{2} + b z \right)^2},\]
\[p_{NB}^{\text{pub}} = \frac{1}{2} - \frac{1}{6} b^2.\]  

Q.E.D.

**Proof of Lemma 3.** Due to the normalisation of consumers to unit, the expected number of consumers that buy the product at the highest price under public information \((p_{L}^{\text{pub}})\) is given by,
\[\gamma_{L}^{\text{pub}} = \Pr (A \text{ wins } | s = (\alpha, \alpha)) \Pr (s = (\alpha, \alpha)) + \Pr (B \text{ wins } | s = (\beta, \beta)) \Pr ((s = (\beta, \beta))
= \frac{1}{4} + \frac{1}{4} b + \frac{1}{12} b^2 - \frac{1}{2} z - b z^2 - \frac{2}{3} b^2 z^3.

Similarly, the expected number of consumers that buy the product at the lowest price \((p_{D}^{\text{pub}})\) is given by
\[\gamma_{D}^{\text{pub}} = \Pr (A \text{ wins } | s = (\beta, \beta)) \Pr (s = (\beta, \beta)) + \Pr (B \text{ wins } | s = (\alpha, \alpha)) \Pr (s = (\alpha, \alpha)
= \frac{1}{4} - \frac{1}{4} b + \frac{1}{12} b^2 + \frac{1}{2} z + b z^2 + \frac{2}{3} b^2 z^3,

and the number of consumers that pay the non-biased price equals
\[\gamma_{NB}^{\text{pub}} = \Pr (A \text{ wins } | s = (\alpha, \beta)) \Pr (s = (\alpha, \beta))
+ \Pr (A \text{ wins } | s = (\beta, \alpha)) \Pr (s = (\beta, \alpha)) + \Pr (B \text{ wins } | s = (\alpha, \beta)) \Pr (s = (\alpha, \beta))
+ \Pr (B \text{ wins } | s = (\beta, \alpha)) \Pr (s = (\beta, \alpha)) = \frac{1}{2} - \frac{1}{6} b^2.

Under non-discrimination the expected number of consumers that pay \(p_{L}^{\text{pub}}, p_{D}^{\text{pub}}\) and \(p_{NB}^{\text{pub}}\) is
\[\gamma_{L, NB}^{\text{pub}} = \frac{1}{4} + \frac{1}{4} b + \frac{1}{12} b^2,\]

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\[ \gamma_{D}^{\text{N.pub}} = \frac{1}{4} - \frac{1}{4} b + \frac{1}{12} b^2, \]
\[ \gamma_{NB}^{\text{N.pub}} = \frac{1}{2} - \frac{1}{6} b^2. \]

Thus, the expected number of inefficient shoppers under public information denoted by \( \text{EIS}^{\text{pub}} \) is given by:
\[ \text{EIS}^{\text{pub}} = (\gamma_{L}^{\text{N.pub}} - \gamma_{L}^{\text{pub}}) = \frac{1}{2} z + b z^2 + \frac{2}{3} b^2 z^3. \quad \text{Q.E.D.} \quad (59) \]

**Proof of Corollary 8.** Using (30) and (44) and the fact that \( z > y \) it follows that
\[ \text{EIS}^{\text{pub}} - \text{EIS} = \frac{1}{2} (z - y) + b (z^2 - y^2) + \frac{2}{3} b^2 (z^3 - y^3) > 0. \quad \text{Q.E.D.} \]

**Proof of Lemma 4.** When the signal observed by firms to each customer is public we may analyse the following scenarios:

<table>
<thead>
<tr>
<th>( s = (i, j) )</th>
<th>( (p_i^\text{pub}, p_j^\text{pub}) )</th>
<th>( \Pr(s = (i, j) \mid l) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s = (\alpha, \alpha) )</td>
<td>( p_{NB}^\text{pub}, p_{NB}^\text{pub} )</td>
<td>( \Pr(s = (\alpha, \alpha) \mid l) = (q(l))^2 )</td>
</tr>
<tr>
<td>( s = (\alpha, \beta) )</td>
<td>( p_{LB}^\text{pub}, p_{LB}^\text{pub} )</td>
<td>( \Pr(s = (\alpha, \beta) \mid l) = q(l) (1 - q(l)) )</td>
</tr>
<tr>
<td>( s = (\beta, \beta) )</td>
<td>( p_{LB}^\text{pub}, p_{LB}^\text{pub} )</td>
<td>( \Pr(s = (\beta, \beta) \mid l) = (1 - q(l))^2 )</td>
</tr>
<tr>
<td>( s = (\beta, \alpha) )</td>
<td>( p_{NB}^\text{pub}, p_{NB}^\text{pub} )</td>
<td>( \Pr(s = (\beta, \alpha) \mid l) = (1 - q(l)) q(l) )</td>
</tr>
</tbody>
</table>

Therefore, expected consumer surplus under public information, denoted by \( \text{ECS}^{\text{pub}} \) is
\[ \text{ECS} = \left( v - p_{NB}^\text{pub} \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - q(l)) q(l) f(l) dl + \int_{-\frac{1}{2}}^{\frac{1}{2}} \max \left( v - p_{LB}^\text{pub}, v - p_{LB}^\text{pub} - l \right) (q(l))^2 f(l) dl \]
\[ + \left( v - p_{DB}^\text{pub} \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} (q(l))^2 f(l) dl + \int_{-\frac{1}{2}}^{\frac{1}{2}} \max \left( v - p_{LB}^\text{pub}, v - p_{LB}^\text{pub} + l \right) (1 - q(l))^2 f(l) dl \]
\[ + \left( v - p_{DB}^\text{pub} \right) \int_{0}^{\frac{1}{2}} (1 - q(l))^2 f(l) dl. \]

Using previous computations and using the fact that \( z = p_{LB}^\text{pub} - p_{DB}^\text{pub} \), after some algebra:
\[ \text{ECS}^{\text{pub}} = v - p_{NB}^\text{pub} \gamma_{NB}^{\text{pub}} - p_{LB}^\text{pub} \gamma_{LB}^{\text{pub}} - p_{DB}^\text{pub} \gamma_{DB}^{\text{pub}} - \frac{1}{4} z^2 - \frac{2}{3} b z^3 - \frac{1}{2} b^2 z^4. \quad (60) \quad \text{Q.E.D.} \]
Proof of Lemma 5.
Given that welfare is equal to industry profit plus consumer surplus one gets,

\[ EW^{pab} = p_L^{pab} \gamma_L^{pab} + p_D^{pab} \gamma_D^{pab} + p_{NB}^{pab} \gamma_{NB}^{pab} \]

\[ + v - p_{NB}^{pab} \gamma_{NB}^{pab} - p_L^{pab} \gamma_L^{pab} - p_D^{pab} \gamma_D^{pab} - \frac{1}{4} z^2 - \frac{2}{3} b z^3 - \frac{1}{2} b^2 z^4. \]

Thus,

\[ EW^{pab} = v - \frac{1}{4} z^2 - \frac{2}{3} b z^3 - \frac{1}{2} b^2 z^4. \]  \hspace{1cm} (61)

Q.E.D.

Proof of Proposition 6. Here we prove analytically that \( EW^{pab} < EW \). Using (36) and (45) and the fact that \( z > y \) it follows that

\[ EW^{pab} - EW = -\frac{1}{4} (z^2 - y^2) - \frac{2}{3} b (z^3 - y^3) - \frac{1}{2} b^2 (z^4 - y^4) < 0. \]

Q.E.D.

References


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