TURNOUT AND POWER SHARING

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Abstract. Different institutional systems determine different mappings from election outcomes to power shares. We study such institutions within a mobilization model as well as in a model with rational voters. Assuming heterogeneity in the cost of voting, the effect of all such institutional differences on turnout depends on the expected closeness of the election: when two parties are expected to have similar support, turnout is higher the closer the system is to a winner take all one; the result is the opposite when one party has a larger expected base. We compare competition effect, size effect and underdog effect under different systems and show the robustness of all comparisons to changes in the number of parties and across models. Turnout is also shown to increase with the number of parties in the proportional system.
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1. Introduction

Voters' participation is an essential component of democracy. Yet the positive analysis of turnout is still far from established and many questions remain. Is it possible to unambiguously characterize the influence of institutional systems on turnout? In particular, does turnout depend in any identifiable way on the mapping from electoral outcomes to power sharing, which obviously varies a lot across systems and may be affected itself by many aspects of an institutional system?

The power of the majority party varies with the degree of separation of powers, the organization of chambers, the assignment of committee chairmanships and institutional rules on agenda setting, allocation of veto powers, and obviously electoral rules.[1] Electoral rules determine the mapping from vote shares to seat shares in a legislature, whereas all the other institutions mentioned above determine the subsequent mapping from seat shares to power shares across parties. Can we identify some general way in which all these institutions affect mobilization efforts by parties and voters' incentives to vote?

Recognizing that turnout depends on individual voters' incentives as well as mobilization efforts by parties, this paper aims to provide a framework in which the parties strategies and individual rational choices can be analyzed together and yield consistent answers to the important questions laid out above. Rather than analyzing institutions one by one, we abstract from all details about how each institution affects power sharing for any electoral outcome, and we study the impact on turnout of the reduced form mapping from vote share to power share.[2] In other words, the results of this paper will all focus on the mapping from vote shares to power shares as summary independent variable in explaining turnout, while the role of individual institutions could be evaluated by referring to their expected impact on that mapping, (e.g. from studies like that of Lijphart (1999) and Powell (2000)).

Having said that the key variable we analyze is the degree of proportionality of influence on policy determination power given electoral outcomes, the results will depend crucially also on the interaction with another key parameter, namely the "expected closeness" of an election. In order to incorporate this in our analysis in the simplest possible way,

[1] See Lijphart (1999) and Powell (2000) for a comprehensive analysis of the impact of political institutions on the degree of proportionality of influence.

[2] See Besley and Persson (2008) for another example in which the modeling decision to compound checks and balances after elections and characteristics of the electoral rules together in a reduced form measure of proportionality of influence has heuristic power.

we assume that voters' preferences over the set of alternatives (candidates or parties or coalitions of parties) are given and common knowledge, and that therefore the only relevant decision by voters is whether to go to vote or not. Each voter is describable in a two-dimensional type space, i.e. with her preferred party and her cost of voting. We first study these questions by fixing the individual benefit of voting and making parties choose mobilization strategies, which affect the distribution of costs of voting (see e.g. Shachar & Nalebuff (1999) for a detailed description of such mobilization efforts). In that model we show that for symmetric priors turnout is higher with a winner take all system, and if the spread between favorite and underdog is sufficiently large it is higher in a system with full proportionality of influence.

The second model that we consider is a rational voter model. This model takes the distribution of voting costs as given, but lets the individual benefit of voting be endogenous to the institutional system. In a fully proportional system the expected marginal benefit of an individual vote is proportional to the marginal change in the vote share determined by the extra vote, whereas in a winner take all system the marginal benefit of a vote is proportional to the probability of that vote being pivotal. Both such marginal benefits decrease as the number of voters increases, but the comparison in a large election depends on the speed of such a reduction in benefits and turnout (size effect) for different ex ante evaluations of the relative strength of parties. Once again, like in the mobilization model, we show that turnout is higher in the winner take all system when the election is perceived to be close ex ante, whereas turnout is higher in the proportional system otherwise.

Levine & Palfrey (2007) studied the behavioral phenomena identified as "size effect, competition effect and underdog effect" in a winner take all system experiment. We are able to compare such effects across systems. The size effect is the decline in turnout as the population gets larger. We show that the size effect is stronger in a proportional system only when the election is expected to be close, but much stronger in a winner-take-all system when the election has a clear favorite. Hence for a large electorate, unless the election is expected to be close, turnout is higher in a proportional system than in a majority system, which matches what we obtain in the mobilization model.

Given the heterogeneity of voting costs, we show that rational voters' decisions determine a partial compensation for the initial asymmetry in party support. In other words, in equilibrium the underdog party always has higher turnout than the leading party, but not enough to bring the election to an expected tie[3]. This partial underdog effect varies with the degree of proportionality of influence induced by the institutional system but is always present. Under some conditions on the distribution of voting costs in the population the underdog effect is greater in a proportional influence system than in a winner-take-all one.

As for the competition effect, we show that turnout increases dramatically as we approach a close election in a majority system, while we have no such dramatic change in a proportional system, which is intuitive because in that case the event of a close election is not as crucial as in a winner take all system.

Even though we conduct the bulk of the analysis for the case of two parties, we show the robustness of all comparisons to changes in the number of parties: in a proportional system the size effect does not depend on the distribution of ex ante support of parties, and we show that the way turnout depends on the size of the electorate does not change with the number of parties either. Hence the comparison with the winner take all system is also unaffected by the number of parties. Finally, we show that in a proportional system turnout increases as the number of parties increases.

We emphasize that our results do not need to invoke any of the standard arguments made about proportional representation, like fairness and representation reasons for turning out.[4] The interaction effect of proportionality of influence and closeness of elections can be explained purely on the basis of rational calculus.

2. General Setup

Consider a polity in which 2 parties (or 2 coalitions of parties) compete for seats in a Parliament. Voters have exogenous political preferences for one or the other, but voting is not compulsory, and hence turnout is uncertain because some or all voters may find voting too costly when compared with the benefit. Each voter v has her own cost of voting c_v, and parties know the distribution of voting costs.

[3]This contrasts with the result by Goree and Grosser (2007), in which there is full compensation and the election is a toss-up. Full compensation occurs if the cost of voting is the same for every agent, as they assume.

[4]For example, Jackman (1987) argued that "minor parties find it difficult to get their candidates elected in highly disproportional systems as their supporters may feel that their votes will be wasted and as a result may be inclined to abstain. PR is a fairer system and where people feel less alienated and are thus more inclined to vote." Our formal results do not need to invoke fairness or representation.

The electorate is composed by a large number n of potential voters, where qn are the set of supporters of party A and $(1-q)n$ are the supporters of party B. In other words, q is the fraction of voters who would vote for A if they turn out, and $(1-q)$ is the fraction of citizens who would vote for B if they turn out.

Denote by a the number of supporters of party A that will actually vote, and correspondingly define b the same way. The vote share for party A then is

$$x \equiv \frac{a}{a+b}.$$

For any vote share x, an institutional system determines power shares $P_A^\gamma(x)$ and $P_B^\gamma(x)$, and these are in general the reduced form components of parties' and voters' utility functions that will determine the incentives to campaign and vote respectively, where γ will be a measure of majority power induced by the institutions.

Turnout will be denoted by

$$T \equiv \frac{a+b}{n}.$$

The different models that we consider below emphasize different channels through which different voting rules or power sharing assumptions affect voting decisions and hence T. We will consider first a mobilization model, to then move to a more elaborate analysis of the rational voter model.

3. Mobilization Model

Assume that all voters have a benefit $\beta > 0$ from voting for their own preferred party, and voter v votes if and only if $\beta \geq c_v$. Since what matters is the net benefit, let's assume without loss of generality that the benefit β is constant across citizens and not affected by anything parties can do, whereas the distribution of costs can be affected by parties' spending.[5] Let us approximate here the large electorate with

[5]As argued by Shachar and Nalebuff (1999), "parties decrease the direct cost of voting, for example they organize volunteers to drive people to the polls, Second, they decrease the cost of becoming informed. third, they increase the cost of not voting by imposing social sanction on those who do not participate." Of course one could equivalently model mobilization efforts by parties as affecting benefits for given costs, saying that aparty's spending makes all it's supporters feel the urgency of the moment, the intensity of the difference between having a ruler of one party or the other, as in Epstein, Morelli and O'Halloran (2008). These two approaches are obviously equivalent conceptually, but in this paper the assumption that mobilization efforts affect primarily the cost side of the equation is more convenient, for reasons that will be clear when both models will be presented.

the unit interval, which is therefore also the support of the distribution of voting costs. Let the distribution of voting costs among supporters of party i (known to both parties) be defined as $F_i(c) = c^{\frac{1}{1+s_i}}$, where $s_i \geq 0$ is the parameter affected by the party's mobilization efforts.

Let s_A, s_B denote the effort/spending/campaigning level by the two parties (spending henceforth), to be determined in equilibrium of a simultaneous move game. Note that without any effort ($s_i = 0\ \forall i$) the distribution is uniform. The spending costs are $l_i(s_i)$, increasing and convex, with $l_i(0) = l_i'(0) = 0,\ l_i'(s) > 0\ \ \forall s > 0,\ \forall i.$

For any spending profile s, the vote share for party A is

$$x(s) = \frac{a(s)}{a(s)+b(s)} = \frac{q\beta^{\frac{1}{1+s_A}}}{q\beta^{\frac{1}{1+s_A}} + (1-q)\beta^{\frac{1}{1+s_B}}} \tag{1}$$

For each institutional setting γ and vote share x, party A has an expected power share $P_A^\gamma(x)$, and $P_B^\gamma(x) = 1 - P_A^\gamma(x)$. When choosing it's spending level, each party maximizes the utility function

$$U_i(s_i, s_{-i}) = P_i^\gamma(x(s_i, s_{-i})) - l_i(s_i).$$

The expected power share function that different parties may have in mind at the time of the spending decision depends on the institutional system: the closer the system is to pure winner-take-all, the steeper the increase of power share when going from a vote share slightly less than 1/2 towards the 1/2 threshold;[6] On the other hand, the closer the system is to a consensus democracy the closer the power shares will be to be linear in the vote shares. Formally, we can capture this institutional determination of power sharing with a simple parameter $\gamma \geq 1$:

$$P_A^\gamma(x) = \begin{cases} \frac{1}{2}(2x)^\gamma & \textit{if}\ \ x < 1/2 \\ 1 - \frac{1}{2}(2(1-x))^\gamma & \textit{if}\ \ x \geq 1/2 \end{cases}$$

and, of course, $P_B^\gamma(x) = 1 - P_A^\gamma(x)$.[7]

$U_i(s_i, s_{-i})$ is continuous in s_i for every $\gamma \geq 1$. The best response $s_i^*(s_{-i})$ is certainly less than s_{-i} when s_{-i} goes to infinity; moreover,

[6] In a winner-take-all system what matters is having the majority of votes, either because the majority of votes translates into obtaining all the seats in the Parliament, or because having the majority in the Parliament suffices to determine policies, without any concession to the minority party in the Parliament. Hence the expected power share is the probability of being the max party.

[7] Note that if $\gamma = 1$ then power is linearly increasing in the vote share, whereas if $\gamma \to \infty$ the institutional system is winner take all.

the best response to $s_{-i} = 0$ is strictly positive[8] and hence an interior equilibrium must exist for every $\gamma \geq 1$.

Given $q \geq \frac{1}{2}$, assume that in equilibrium $x(s^*) \geq 1/2$, so that we can use just one of the two pieces of the power share function; then the validity of the assumption will be confirmed by the solution, since we prove that in equilibrium, for every $q \in [1/2, 1)$ and for every $\gamma > 1$ the two parties spend equal amounts in mobilization efforts.

Lemma 1. *The equilibrium spending level $s = s_A = s_B$ solves*

$$l'(s)(1+s)^2 = \frac{\gamma q\,(-\ln\beta)}{2}\,(2(1-q))^{\gamma}\,. \tag{2}$$

Proof. See Appendix. □

Having solved for the equilibrium spending level for every q and every $\gamma \geq 1$, it is possible to compute turnout, and we can conclude that

Proposition 2. *(I) There exists $\hat{q} \in (1/2, 1)$ such that for every $q \in (1/2, \hat{q})$ turnout is maximal for some intermediate $\gamma^*(q) > 1$;*
(II) When $q = 1/2$ turnout is strictly increasing in γ; $\gamma^*(q)$ converges to infinity as q converges to $1/2$;
(III) On the other hand, turnout is maximal with $\gamma = 1$ for every $q > \hat{q}$.

Proof. See the appendix. □

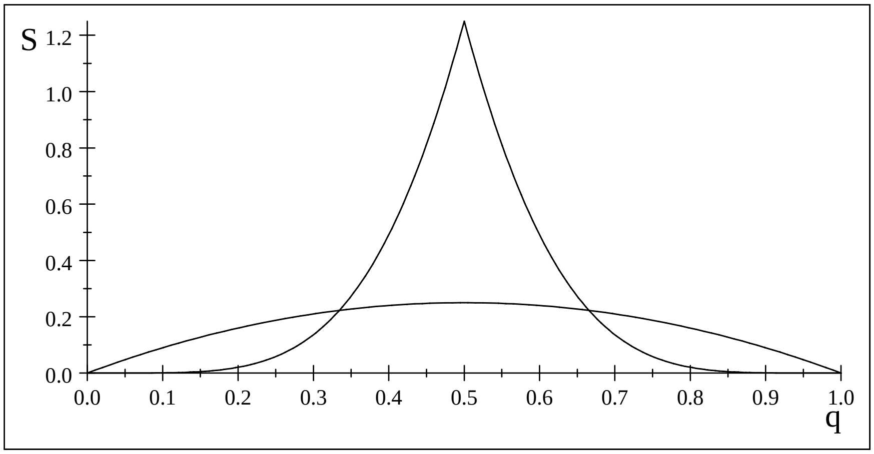
Proposition 2 implies that if we compare turnout for $\gamma = 1$ (pure proportionality) and a high γ that approximates a winner take all system, the result depends crucially on how close the election is expected to be:

Corollary 3. *There exists $q^* \in (1/2, \hat{q})$ such that turnout is higher with a winner take all system than with a pure proportional system if and only if $q < q^*$.*

This result is also displayed in the picture below, which represents party spending as a function of the closeness of the election q, both for $\gamma = 1$ (PR) and for $\gamma = 5$, approximating a majority rule (MR) system.

[8] This is because it can be shown that the marginal utility of spending at 0,0 is

$$-\ln\beta * \gamma q(1-q)(2(1-q))^{\gamma-1} > 0.$$



We will see that this result is very similar to the main comparative result in the rational voter model that will follow.

A few words about the choice of specific functional forms: (1) All the above results of the mobilization model are robust to changes in the specific functional form of the power function. For example, it is possible to check that if we used a power function similar to a contest success function typically employed in the contest literature (see e.g. Hirschleifer (1989)), the qualitative results would be unchanged. (2) The choice of functional form for the distribution of costs of voting could also be changed to many others, but we chose this because it also works well for computational purposes in the rational voter model, to which we now turn.

4. Pivotal Voter Model

In section 3 voters' decision to turn out or not was reduced to a comparison between a fixed benefit and the individual cost of voting, and the focus was on the parties' strategic attempts to affect this calculation through all kinds of mobilization efforts and campaign spending. In this section we do the opposite: we fix the distribution of voting costs and parties mobilization behavior, and we focus instead on turnout decisions by rational voters who are able to compute the expected benefit of voting under the different institutions considered in this paper.

We return here to a large but finite population, but with population uncertainty. There are n citizens distributed as a Poisson distribution with mean N,

$$n \sim \frac{e^{-N} (N)^n}{n!}$$

Citizens have to choose to vote for party A, party B, or abstain.

We look only at symmetric equilibria where players of the same type choose the same strategy. Types differ along two lines: in their preference for party A or B and in their voting cost. Any citizen has a chance $q \in (0,1)$ of being a supporter of party A and a chance $(1-q)$ of supporting party B and has a cost drawn from a distribution with cdf

$$F(c), \quad c \in [0,1]$$

The benefit to the voter if his preferred party obtains all the power is normalized to one, hence c can be seen as a cost benefit ratio. If a share α of A types vote for A and a share β of B types vote for B, the expected turnout T is

$$T := q\alpha + (1-q)\beta$$

Assume that $q \leq 1/2$ so A is always the party with less supporters, and this is without loss of generality as $q > 1/2$ can be obtained by switching the party labels.

We look for a symmetric equilibrium in which all voters of type A with a cost below a threshold c_α vote for type A and voters of type B with a cost below c_β vote for B. So type A citizens vote for A with chance $\alpha = F(c_\alpha)$ and type B citizens vote for B with chance $\beta = F(c_\beta)$.

In any (α, β) symmetric strategy profile, the expected marginal benefit of voting B_i must be equal to the cutoff cost of voting. Hence the equilibrium conditions can be written as

$$B_i^A(\alpha,\beta) = F^{-1}(\alpha), \quad B_i^B(\alpha,\beta) = F^{-1}(\beta)$$

We compare two systems: majority rule (MR: $i = M$) and proportional representation (PR: $i = P$).[9,10]

The statements below are made for a large enough population, namely they are true for every N above a given $\overline{N}$.

[9]These two systems correspond to the extreme cases of $\gamma = 1$ and $\gamma = \infty$ in section 3.

[10]Recall that, even if we use the language PR and MR almost everywhere below, the interpretation is not restricted to electoral rules, as explained in the introduction. Two countries with the same electoral rule can have very different mappings from electoral outcomes to power shares, and this is the summary or reduced form variable that we are interested in and that affects turnout.

4.1. **Majority Rule (MR).** In the MR system the expected marginal benefit of voting B_M^A is the chance of being pivotal for a type A citizen

$$B_M^A = \sum_{k=0}^{\infty} \left(\frac{e^{-qN\alpha} (Nq\alpha)^k}{k!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q) N\beta)^k}{k!} \right) \left(\frac{1}{2} \right) \left(1 + \frac{(1-q) N\beta}{k+1} \right)$$

namely the chance that an A citizen by voting either makes a tie and wins the coin toss or breaks a tie where it would have lost the coin toss. Likewise, for the type B citizens

$$B_M^B = \sum_{k=0}^{\infty} \left(\frac{e^{-qN\alpha} (Nq\alpha)^k}{k!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q) N\beta)^k}{k!} \right) \left(\frac{1}{2} \right) \left(1 + \frac{qN\alpha}{k+1} \right)$$

Lemma 4. *There exists an equilibrium in the MR system. For uniqueness it suffices that F is weakly concave.*

Proof. See Appendix. □

4.2. **Proportional Representation (PR).** In a PR system the share of power is proportional to the vote share obtained in the election. So if the (a, b) are the absolute number of votes for each party, the power of parties A and B would be respectively

$$\left(\frac{a}{a+b}, \frac{b}{a+b} \right)$$

We assume that if nobody votes, power is shared equally, namely

$$\frac{a}{a+b} = \frac{b}{a+b} = \frac{1}{2} \quad for \quad a = b = 0$$

In a PR system the expected marginal benefit of voting B_P^A is the expected increase in the vote share for the preferred party induced by a single vote, namely

$$\begin{aligned} B_P^A &= \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\begin{array}{c} \left(\frac{e^{-qN\alpha}(qN\alpha)^a}{a!} \right) \left(\frac{e^{-(1-q)N\beta}((1-q)N\beta)^b}{b!} \right) \\ \left(\frac{a+1}{a+b+1} - \frac{a}{a+b} \right) \end{array} \right) \\ B_P^B &= \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\begin{array}{c} \left(\frac{e^{-qN\alpha}(qN\alpha)^a}{a!} \right) \left(\frac{e^{-(1-q)N\beta}((1-q)N\beta)^b}{b!} \right) \\ \left(\frac{b+1}{a+b+1} - \frac{b}{a+b} \right) \end{array} \right) \end{aligned}$$

Lemma 5. *The marginal benefit of voting in PR has the closed form*

$$\begin{aligned} B_P^A &= \frac{(1-q)\,\beta}{NT^2} - e^{-NT}\left(\frac{((1-q)\,\beta)^2 - (q\alpha)^2 + (1-q)\,\beta\frac{1}{N}}{2T^2}\right) \\ B_P^B &= \frac{q\alpha}{NT^2} + e^{-NT}\left(\frac{((1-q)\,\beta)^2 - (q\alpha)^2 - q\alpha\frac{1}{N}}{2T^2}\right) \end{aligned}$$

Proof. See Appendix. □

Using this lemma, the sum of the marginal benefits for the two types is

$$B_P^A + B_P^B = \frac{1}{NT}\left(1 - \frac{e^{-NT}}{2}\right)$$

which decreases like the inverse of the expected number of voters N when the latter is large.

Lemma 6. *In the PR system an equilibrium always exists and the equilibrium is unique.*

Proof. See Appendix. □

Proposition 7. *In the PR system the equilibrium has the following properties:*
Underdog Property: the party with less supporters gets a lower number of votes, but turns out in higher percentage than the other party

$$q < 1/2 \quad \Longrightarrow \quad q\alpha < (1-q)\,\beta, \quad \alpha > \beta$$

Size Effect: larger population displays lower turnout

$$\frac{dT}{dN} < 0$$

Proof. See Appendix. □

4.3. **Comparison of MR with PR.** The turnouts and underdog effects in the two models compare as follows.

Proposition 8. *Turnout is larger in PR unless the election is expected to be close*

$$\begin{aligned} q &\neq 1/2 \quad \Longrightarrow \quad T_{MR} < T_{PR} \\ q &= 1/2 \quad \Longrightarrow \quad T_{MR} > T_{PR} \end{aligned}$$

In both models the leading party obtains the majority but the underdog party has a higher turnout of its supporters. The underdog effects in the two models compare as follows

$$\frac{\alpha_P F^{-1}(\alpha_P)}{\beta_P F^{-1}(\beta_P)} > \frac{\alpha_M F^{-1}(\alpha_M)}{\beta_M F^{-1}(\beta_M)}$$

Proof. See Appendix. □

The intuition behind this result relies on how fast the marginal benefit of voting decreases in the two models as the electorate gets larger. This benefit in MR drops asymptotically as

$$B_M \sim \frac{e^{-\left(q-\frac{1}{2}\right)N}}{\sqrt{N}}$$

So, B_M has two different rates of convergence regimes: it decreases exponentially for any $q \neq 1/2$ and for $q = \frac{1}{2}$ it decreases at the much slower algebraic rate of $N^{-1/2}$. The intuition is that since we have only partial compensation from the underdog effect, then for any $q \neq 1/2$ the majority party is always the more likely side to win. Hence the chance of a tied election, which is what drives rational voters to turn out, is much smaller than in the case $q = 1/2$ for any population size N. The two rates of convergence derived above are not particular to the Poisson uncertainty of this model.[11]

The marginal benefit from voting in PR drops asymptotically at the intermediate rate of

$$B_P \sim \frac{1}{N}$$

This rate is independent of q as in PR the chance of being the pivotal voter, i.e. the event of a tied election, has no special relevance any longer.

It is perhaps intuitive that MR, unlike PR, should have two quite different rates of convergence regimes. On the other hand, only explicit computation can determine that the rate of convergence in PR is indeed intermediate between the two rates of convergence in MR.

4.4. **Parametric Example.** We can solve for the parametric example in which the cost distribution belongs to the family

$$c \in [0,1], \quad F(c) = c^{1/z}$$

[11]Herrera & Martinelli (2006) analyze an MR election without population uncertainty. They introduce aggregate uncertainty in a different way, which allows to obtain a closed form for the chance of being pivotal, namely

$$\frac{(a+b)!}{2^{a+b+1}a!b!}$$

As it can be seen using Stirling's approximation, that marginal benefit for large a and b has exactly the square root decline on the diagonal $(a = b)$ and the exponential decline off the diagonal $(a = \omega b,\ \omega \neq 1)$.

The PR system is

$$\beta_P = \left(\frac{q}{1-q}\right)^{\frac{1}{z+1}} \alpha, \qquad \alpha^z = \frac{1}{N} \frac{(1-q)\,\beta_P}{(q\alpha + (1-q)\,\beta_P)^2}$$

which gives the closed form solution

$$\begin{cases} \alpha = \left(\frac{1}{N} \frac{(1-q) q^{\frac{1}{z+1}} (1-q)^{\frac{1}{z+1}}}{\left(q(1-q)^{\frac{1}{z+1}} + (1-q) q^{\frac{1}{z+1}} \right)^2} \right)^{\frac{1}{z+1}} \\ \beta_P = \left(\frac{1}{N} \frac{q}{1-q} \frac{(1-q) q^{\frac{1}{z+1}} (1-q)^{\frac{1}{z+1}}}{\left(q(1-q)^{\frac{1}{z+1}} + (1-q) q^{\frac{1}{z+1}} \right)^2} \right)^{\frac{1}{z+1}} \end{cases}$$

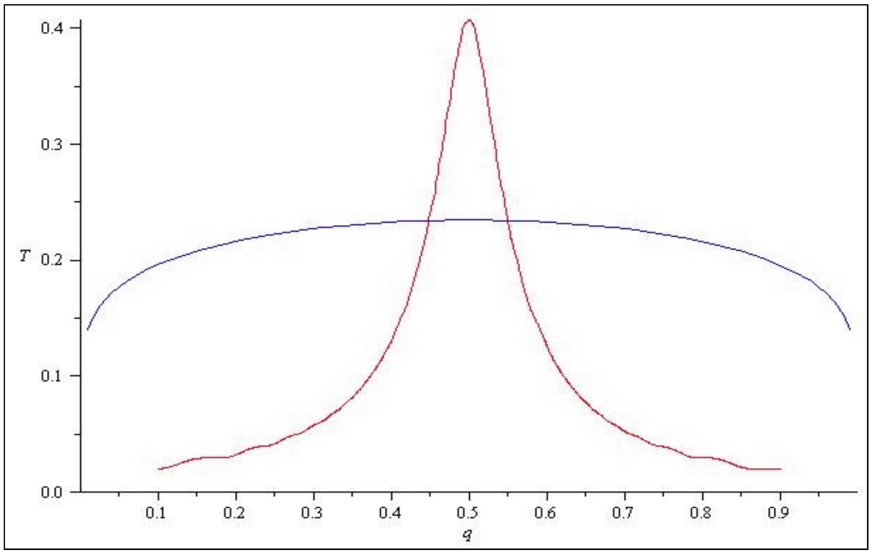
The MR system is

$$\begin{aligned} \beta_M &= \left(\frac{q}{1-q}\right)^{\frac{1}{2z+1}} \alpha \\ \alpha^z &= \frac{e^{-N\left(\sqrt{(1-q)\beta_M} - \sqrt{q\alpha}\right)^2}}{\sqrt{N}} \left(\frac{\sqrt{q\alpha} + \sqrt{(1-q)\,\beta_M}}{4\sqrt{\pi}\,(q\,(1-q)\,\alpha\beta_M)^{1/4}} \right) \end{aligned}$$

4.5. **Numerical Results.** We present here some numerical results on turnout. We set

$$N = 3000, \quad z = 5$$

In the following plot we show the turnout in the two systems as a function of q. As you can see T_M spikes at $q = 1/2$ and around that value is larger than T_P.



To have an idea of the magnitudes, for $q = 1/3$ we have in the PR system

$$\alpha = 24.8\% > \beta_P = 22\%, \quad T_P = 23\%$$

and in the MR system

$$\alpha_M = 7.1\% > \beta_M = 6.7\%, \quad T_M = 6.8\%$$

Turnout in MR spikes up and surpasses turnout in PR when the election becomes close. So for $q = 1/2$ we have

$$T_M = 40.9\% > 23.5\% = T_P$$

As for the competition effect, it seems intuitive that it should be stronger for MR: as party support becomes more even turnout increases dramatically in MR. Yet, as it is clear from the picture this is only for values of q close to 1/2. If the party support is very uneven, say $q = 10\%$, the impact of an increase in q on turnout is very small in MR and may be higher in PR. than MR.

As for the underdog effect, in this example with this cost function we have

$$\frac{\alpha_P F^{-1}(\alpha_P)}{\beta_P F^{-1}(\beta_P)} > \frac{\alpha_M F^{-1}(\alpha_M)}{\beta_M F^{-1}(\beta_M)} \implies$$
$$\frac{\alpha_P^{z+1}}{\beta_P^{z+1}} > \frac{\alpha_M^{z+1}}{\beta_M^{z+1}} \implies \frac{\alpha_P}{\beta_P} > \frac{\alpha_M}{\beta_M}$$

so the underdog effect, by this measure, is larger in PR than in MR.

In sum, there exist distributions of voting costs, like the one also used in the mobilization model, such that the underdog effect is higher under PR. Also the competition effect can be higher under PR, but only when the starting point in terms of the distribution of party supporters is sufficiently asymmetric. On the other hand, the size effect is higher under PR only when the distribution of party supporters is close enough to symmetric.

5. Many Parties

We extend the previous results and intuition to the case of more than two parties. We compute explicitly here just the case of three parties. For more than three parties the derivations are analogous.

Define

$$\begin{aligned} A \quad &: \quad = \alpha q_A N, \quad B := \beta q_B N, \quad C := \gamma q_C N \\ \text{with:} \quad & \quad q_A + q_B + q_C = 1 \end{aligned}$$

The marginal benefit for party A for instance is

$$B_P^A = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \left(\frac{e^{-A} A^a}{a!} \right) \left(\frac{e^{-B} B^b}{b!} \right) \left(\frac{e^{-C} C^c}{c!} \right) \left(\frac{a+1}{a+b+c+1} - \frac{a}{a+b+c} \right)$$

We assume again that if nobody votes power is shared equally, namely

$$\frac{a}{a+b+c} = 1/3 \quad \text{for } a = b = c = 0$$

Lemma 9. *The marginal benefit has the closed form*

$$B_P^A = \left(1 - \frac{A}{A+B+C}\right) \frac{1 - e^{-(A+B+C)}}{A+B+C} + \left(\frac{A}{A+B+C} - \frac{1}{3}\right) e^{-(A+B+C)}$$

Proof. See Appendix. □

The analog for the parties B or C is straightforward.

Proposition 10. *(I) The comparison between turnout in PR and MR is unchanged. (II) If parties are symmetric, turnout in PR increases as the number of parties increases.*

Proof. A similar calculation gives the analogous result for r parties

$$B_P^A(r) = \left(\begin{array}{c} \left(1 - \frac{A}{A+B+C+\ldots+r}\right) \frac{1-e^{-(A+B+C+\ldots+r)}}{A+B+C+\ldots+r} \\ + \left(\frac{A}{A+B+C+\ldots+r} - \frac{1}{r}\right) e^{-(A+B+C+\ldots+r)} \end{array} \right)$$

For large enough N, B_P^A approximates to

$$\begin{aligned} B_P^A &\simeq \left(1-\frac{A}{A+B+C+...+r}\right)\frac{1}{A+B+C+...+r} \\ &= \left(\frac{\beta q_B+\gamma q_C+..}{(\alpha q_A+\beta q_B+\gamma q_C+..)^2}\right)\frac{1}{N} \end{aligned}$$

so the benefit still decreases as N^{-1}, which implies a higher turnout than in MR except in the case when the two parties in MR have the same ex-ante support: $q=1/2$.

For r parties with equal ex-ante support we have

$$q_A=q_B=q_C=...=q_r=1/r \quad \Longrightarrow \quad \alpha=\beta=\gamma=...$$

the first order condition for a party becomes

$$\left(1-\frac{1}{r}\right)\frac{1-e^{-\alpha_r N}}{\alpha_r N}\approx\left(1-\frac{1}{r}\right)\frac{1}{\alpha_r N}=F^{-1}(\alpha_r)$$

so the turnout for that party α_r increases in r. Overall turnout increases too as in this symmetric case we have.

$$T_r=\alpha_r$$

□

The fact that turnout increases with more parties is confirmed by the general fact that smaller parties obtain a higher turnout. The intuition for that follows from the following two observations. First, fixing the number of votes for the other parties z, the vote share increase for party A is

$$\left(\frac{a+1}{a+z+1}-\frac{a}{a+z}\right)=\left(\frac{a^2+a}{z}+2a+1+z\right)^{-1}$$

which is larger for smaller values of the random variable a. Second, a smaller party (with a smaller q_A), assigns in the marginal benefit B_P^A a larger Poisson weight $\left(\frac{e^{-A}A^a}{a!}\right)$ to small values of a.

6. Concluding Remarks and Directions for Future Research

In this paper we have shown that turnout of rational voters, for given distributions of partisan voters and voting costs, depends on the degree of proportionality of influence in the institutional system in the same way as when the turnout is mostly determined by mobilization efforts by parties. In both models, we have shown that turnout is higher in a winner take all system if the initial distribution of partisan voters is symmetric, whereas a more proportional system induces higher turnout otherwise. We have been able to compare underdog effect and size effect for relevant parameter values, and all the comparative results extend to the case in which a proportional system induces the existence of many parties.

Even though the number of parties is exogenous in the paper, the fact that the comparative results in terms of turnout *do not* depend on the number of parties under PR is reassuring, and makes the (hard) extension to endogenous party formation perhaps unnecessary. In light of the robustness results on the number of parties, even the extension to a multistage game in which the parties play some kind of legislative bargaining game after the election is not likely to generate any significant difference in terms of our main comparative results.

One theoretical extension that instead we aim to pursue is the following: what happens if we combine the two models we have studied? To be specific, what happens if we assume that when parties choose their mobilization strategies they expect voters to compute their benefit of voting rationally as a function of the first stage mobilization efforts, rather than assuming a fixed benefit of voting? Could the equal spending result of the mobilization model be robust to this extension, or should we expect a change in some direction? In other words, when voters and parties are all players in the game, are their strategies complements or substitutes in the determination of turnout, given that when they are studied in isolation they determine the same comparative result?

Another theoretical question for future research is about mixed systems. Even though the comparison between the two extreme institutional systems is the same in the models considered here, it is possible, for some distributions of partisan voters, that turnout is maximal for some intermediate degree of proportionality of influence. This is definitely the case in the mobilization model, as one can see from proposition 1, but it has not been technically feasible to verify this possibility in the rational voter model. For the results to determine precise testable predictions it would be nice to characterize turnout incentives in mixed systems, because if the mapping from the degree of proportionality of influence to turnout is non monotonic, then we have to separate the prediction for close elections from that for asymmetric elections, since the expected sign of the coefficient of the proportionality variable depends on the initial conditions.

Beside the intrinsic value of the theoretical results, the findings of this paper could be useful for future empirical as well as experimental research. For example, if one focuses on voting rules, the empirical evidence on turnout in national elections (see e.g. Powell (1980, 1986), Crewe (1981), Jackman (1987) and Jackman and Miller (1995), Blais and Carthy (1990) and Franklin (1996)) all conclude that, everything else being equal, turnout is lower in plurality and majority elections than under Proportional Representation.[12] On the other hand, experimental evidence (see Schram and Sonnemans (1996)) display the opposite finding. We have shown that these seemingly inconsistent findings are instead perfectly reconcilable, since the experimental design employed symmetry in the number of supporters for different parties – the case in which indeed we have shown that we should expect higher turnout under a winner take all system. Future experimental investigations should employ different treatments, allowing for the possibility of asymmetric distributions of partisan supporters and varying the degree of power proportionality. Similarly, we believe that the empirical analysis should be extended beyond electoral rules, since there are many other institutional details that affect the degree of proportionality of power as a function of the allocations of seats determined by the vote shares and the electoral formula. Finally, even the prediction that turnout should increase in the number of parties could be tested experimentally as well as on the existing field data.

7. Appendix

Proof of Lemma 1. For Party A we have

$$\begin{aligned}\frac{\partial x(s)}{\partial s_A} &= \frac{\partial}{\partial s_A}\left(1-\frac{(1-q)\beta^{\frac{1}{1+s_B}}}{q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}}\right)\\ &= \left(-\frac{\ln\beta}{(1+s_A)^2}\right)\frac{\left(q\beta^{\frac{1}{1+s_A}}\right)\left((1-q)\beta^{\frac{1}{1+s_B}}\right)}{\left(q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}\right)^2}\end{aligned}$$

[12]The standard caveat is that cross sectional studies are not to be considered conclusive evidence, because of the small sample size and few data points, cultural and idiosyncratic characteristics that are difficult to control for, as emphasized in Acemoglu (2005).

The first order condition for party A is

$$\begin{aligned} l'_A(s_A) &= \gamma(2(1-x))^{\gamma-1}\left(-\frac{\ln\beta}{(1+s_A)^2}\right)\frac{\left(q\beta^{\frac{1}{1+s_A}}\right)\left((1-q)\beta^{\frac{1}{1+s_B}}\right)}{\left(q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}\right)^2} \\ &= \frac{\gamma(-\ln\beta)}{(1+s_A)^2}\left(2\frac{(1-q)\beta^{\frac{1}{1+s_B}}}{q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}}\right)^{\gamma-1}\frac{\left(q\beta^{\frac{1}{1+s_A}}\right)\left((1-q)\beta^{\frac{1}{1+s_B}}\right)}{\left(q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}\right)^2} \end{aligned}$$

For B we have

$$\begin{aligned} \frac{\partial x(s)}{\partial s_B} &= \frac{\partial}{\partial s_B}\left(\frac{q\beta^{\frac{1}{1+s_A}}}{q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}}\right) \\ &= -\left(-\frac{\ln\beta}{(1+s_B)^2}\right)\frac{\left(q\beta^{\frac{1}{1+s_A}}\right)\left((1-q)\beta^{\frac{1}{1+s_B}}\right)}{\left(q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}\right)^2} \end{aligned}$$

and hence the FOC is

$$\begin{aligned} l'_B(s_B) &= \gamma(2(1-x))^{\gamma-1}\left(-\frac{\ln\beta}{(1+s_B)^2}\right)\frac{\left(q\beta^{\frac{1}{1+s_A}}\right)\left((1-q)\beta^{\frac{1}{1+s_B}}\right)}{\left(q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}\right)^2} \\ &= \frac{\gamma(-\ln\beta)}{(1+s_B)^2}\left(2\frac{(1-q)\beta^{\frac{1}{1+s_B}}}{q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}}\right)^{\gamma-1}\frac{\left(q\beta^{\frac{1}{1+s_A}}\right)\left((1-q)\beta^{\frac{1}{1+s_B}}\right)}{\left(q\beta^{\frac{1}{1+s_A}}+(1-q)\beta^{\frac{1}{1+s_B}}\right)^2} \end{aligned}$$

Taking the ratio of the two FOCs we have

$$l'_A(s_A)(1+s_A)^2 = l'_B(s_B)(1+s_B)^2$$

This implies $s_A = s_B$ since the function $l'(s)(1+s)^2$ is increasing hence injective.

In sum, we have an equal spending solution $s_A = s_B$ which solves

$$l'(s_A)(1+s_A)^2 = \frac{\gamma q(-\ln\beta)}{2}(2(1-q))^{\gamma}$$

□

Proof of Proposition 2. (II) When $q = 1/2$ the equilibrium condition is

$$l'(s)(1+s)^2 = \frac{\gamma(-\ln\beta)}{4}$$

so that increasing γ, i.e., approaching a winner take all system, increases spending and turnout, because it increases the marginal benefit of spending.

(I) and (III): When $q > 1/2$ we have

$$l'(s)(1+s)^2 = \frac{\gamma q\,(-\ln\beta)}{2}\,(2(1-q))^{\gamma}$$

The LHS is increasing in the effort, the RHS is independent of the effort and increases in γ if and only if

$$\gamma < \frac{1}{\ln\left(\frac{1}{2(1-q)}\right)} \tag{3}$$

which is satisfied for q close to $1/2$ and is violated for sufficiently high q. Consider $\hat{q}$ that solves

$$\ln\left(\frac{1}{2(1-q)}\right) = 1 \quad \Longrightarrow \quad \hat{q} \simeq 81.6\%$$

then for every $q \geq \hat{q}$ condition (3) cannot be satisfied, and hence pure proportionality maximizes turnout. $\square$

Proof of Lemma 4. For N large we can use Myerson's approximation and obtain

$$\begin{aligned} B_M^A &\simeq \left(\begin{array}{c} \frac{e^{-N\left(q\alpha+(1-q)\beta-2\sqrt{q(1-q)\alpha\beta}\right)}}{\sqrt{N}} \\ \frac{\sqrt{q\alpha}+\sqrt{(1-q)\beta}}{4\sqrt{\pi}(q(1-q)\alpha\beta)^{1/4}}\frac{1}{\sqrt{q\alpha}} \end{array}\right) = F^{-1}(\alpha) \\ B_M^B &\simeq \left(\begin{array}{c} \frac{e^{-N\left(q\alpha+(1-q)\beta-2\sqrt{q(1-q)\alpha\beta}\right)}}{\sqrt{N}} \\ \frac{\sqrt{q\alpha}+\sqrt{(1-q)\beta}}{4\sqrt{\pi}(q(1-q)\alpha\beta)^{1/4}}\frac{1}{\sqrt{(1-q)\beta}} \end{array}\right) = F^{-1}(\beta) \end{aligned}$$

which yields

$$\sqrt{q\alpha}F^{-1}(\alpha) = \sqrt{(1-q)\,\beta}F^{-1}(\beta)$$

Since the function $\sqrt{x}F^{-1}(x)$ is increasing we can define the function

$$\beta := \beta_M(\alpha)$$

where $\beta_M : [0,1] \longrightarrow [0,1]$ is an increasing differentiable function with $\beta_M(0) = 0$. Hence $F^{-1}(\alpha)$ increases from 0 to 1. Hence the function $B_M^A(\alpha, \beta_M(\alpha))$ is continuous in α, so it remains to that B_M^A is decreasing from infinity. Let's define

$$g := \sqrt{q\alpha}, \quad h := \sqrt{(1-q)\,\beta_M(\alpha)}$$

we have

$$\alpha \in (0,1] \quad \Longrightarrow \quad g < h$$

and for any fixed N we have

$$\lim_{\alpha\to 0}\frac{e^{-N(h-g)^2}}{\sqrt{N}}\left(\frac{g+h}{4\sqrt{\pi}\sqrt{hg}}\right)\frac{1}{g}>\lim_{\alpha\to 0}\frac{e^{-N(h-g)^2}}{\sqrt{N}}\left(\frac{2}{4\sqrt{\pi}\sqrt{hg}}\right)=\infty$$

For $\alpha=1$ we have $h>g=\sqrt{q}$, so for all N above a certain value we have.

$$\left(\frac{e^{-N(h-g)^2}}{\sqrt{N}}\right)\left(\frac{g+h}{4\sqrt{\pi}\sqrt{gh}}\frac{1}{g}\right)<F^{-1}(1)=1$$

which proves existence of a solution. For uniqueness it is left to show that the B_M^A is decreasing in α, namely that the following quantity is negative

$$\begin{aligned}
&= \frac{d}{dg}\left(\left(\frac{e^{-N(h-g)^2}}{\sqrt{N}}\right)\left(\frac{g+h}{4\sqrt{\pi}\sqrt{hb}}\frac{1}{g}\right)\right)\\
&= \left(\frac{e^{-N(h-g)^2}}{\sqrt{N}}\right)\left(-2N(h-g)\frac{d(h-g)}{dh}\left(\frac{g+h}{4\sqrt{\pi}\sqrt{gh}}\frac{1}{g}\right)+\frac{d}{dh}\left(\frac{g+h}{4\sqrt{\pi}\sqrt{gh}}\frac{1}{g}\right)\right)
\end{aligned}$$

For large N this derivative will be negative if and only if

$$\frac{d(h-g)}{da}=\frac{\sqrt{1-q}}{\sqrt{q}}\frac{d\beta'}{d\alpha'}-1>0$$

where we define

$$\begin{aligned}
\beta' &:= \sqrt{\beta},\quad \alpha' := \sqrt{\alpha}\\
G(\alpha') &:= \alpha' F^{-1}\left((\alpha')^2\right)=\sqrt{\alpha}F^{-1}(\alpha)
\end{aligned}$$

we have

$$\sqrt{1-q}G(\beta')=\sqrt{q}G(\alpha')\quad\Longrightarrow\quad\frac{\sqrt{1-q}}{\sqrt{q}}\frac{d\beta'}{d\alpha'}=\frac{G'(\alpha')}{G'(\beta')}$$

So we need G' to be increasing

$$G'(\alpha')=\frac{d}{d\alpha}\left(\sqrt{\alpha}F^{-1}(\alpha)\right)\frac{d\alpha}{d\alpha'}=2\frac{d}{d\alpha}\left(\alpha F^{-1}(\alpha)\right)$$

so it suffices for $\alpha F^{-1}(\alpha)$ to be weakly convex, so $F(\alpha)$ weakly concave is sufficient. □

Proof of Lemma 5. Call the expected number of voters for each party $R:=qN\alpha$, $S:=(1-q)N\beta$ we have

$$B_P^A=e^{-R-S}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\left(\frac{R^a}{a!}\right)\left(\frac{S^b}{b!}\right)\left(\frac{a+1}{a+b+1}-\frac{a}{a+b}\right)$$

By differentiating and integrating the summands and inverting the series and integral operators we have

$$\begin{aligned}\sum_{b=0}^{\infty}\frac{S^b}{b!}\frac{a}{a+b} &= \frac{a}{S^a}\sum_{b=0}^{\infty}\int_0^S\frac{d}{dr}\left(\frac{1}{b!}\frac{r^{a+b}}{a+b}\right)dr=\frac{a}{S^a}\int_0^S\sum_{b=0}^{\infty}\left(\frac{1}{b!}r^{a+b-1}\right)dr\\ &= \begin{cases}\frac{a}{S^a}\int_0^S r^{a-1}e^r dr & \text{for } a\geq 1\\ 1/2 & \text{for } a=0\end{cases}\end{aligned}$$

and

$$\sum_{b=0}^{\infty}\frac{S^b}{b!}\frac{a+1}{a+b+1}=\frac{a+1}{S^{a+1}}\int_0^S r^a e^r dr$$

By inverting the series and integral operators again in the series over a, we have

$$\begin{aligned}B_P^A &= e^{-R-S}\left(\sum_{a=0}^{\infty}\frac{R^a}{a!}\left(\frac{a+1}{S^{a+1}}\int_0^S r^a e^r dr\right)-\sum_{a=1}^{\infty}\frac{R^a}{a!}\left(\frac{a}{S^a}\int_0^S r^{a-1}e^r dr\right)-\frac{1}{2}\right)\\ &= e^{-R-S}\left(\int_0^S\left(\begin{array}{c}\frac{1}{S}\left(\sum_{a=0}^{\infty}\frac{\left(\frac{R}{S}r\right)^a}{a!}+\sum_{a=1}^{\infty}\frac{\left(\frac{R}{S}r\right)^a}{(a-1)!}\right)\\ -\frac{R}{S}\sum_{a=1}^{\infty}\frac{\left(\frac{R}{S}r\right)^{a-1}}{(a-1)!}\end{array}\right)e^r dr-\frac{1}{2}\right)\\ &= e^{-R-S}\left(\frac{1}{S^2}\int_0^S e^{\left(1+\frac{R}{S}\right)r}\left(S-RS+Rr\right)dr-\frac{1}{2}\right)\\ &= e^{-R-S}\left(\frac{1-R}{S}\left(\frac{e^{S+R}-1}{\left(1+\frac{R}{S}\right)}\right)+\frac{R}{S^2\left(1+\frac{R}{S}\right)^2}\int_0^{S+R}e^r r\,dr-\frac{1}{2}\right)\\ &= \frac{S}{(R+S)^2}-\frac{e^{-(R+S)}}{(R+S)^2}\frac{S^2-R^2+S}{2}\end{aligned}$$

and by symmetry

$$B_P^B(R,S)=B_P^A(S,R)$$

□

Proof of Lemma 6. For every $\alpha>0,\beta>0$ as N becomes large both B_P^A and B_P^B tend to zero. Hence, the cost side of the equation shows that the only possible solution to the system $\left(B_P^A=F^{-1}(\alpha),B_P^B=F^{-1}(\beta)\right)$ as $N\longrightarrow\infty$ is $(\alpha=0,\beta=0)$. Therefore, summing the two equations of the PR system

$$\frac{1}{NT}\left(1-\frac{e^{-NT}}{2}\right)=F^{-1}(\alpha)+F^{-1}(\beta)$$

since the RHS goes to zero and the LHS along with it, which means that NT must go to infinity. So for N large the PR system reduces to

$$\frac{(1-q)\,\beta}{NT^2} = F^{-1}(\alpha)\,, \quad \frac{q\alpha}{NT^2} = F^{-1}(\beta)$$

the exponential terms e^{-NT} vanish faster than the hyperbolic terms. So for N large the system gives

$$\begin{aligned} q\alpha F^{-1}(\alpha) &= (1-q)\,\beta F^{-1}(\beta) \\ q &< 1/2 \iff \alpha > \beta \end{aligned}$$

Since $xF^{-1}(x)$ is increasing we can define

$$\beta := \beta_P(\alpha)$$

where $\beta_P(\alpha) : [0,1] \longrightarrow [0,1]$ is an increasing differentiable function with $\beta_P(0) = 0$. We now reduced the PR system to one equation

$$B_P^A := \frac{(1-q)\,\beta_P(\alpha)}{NT^2} = F^{-1}(\alpha)$$

which we now will show has only one solution. The cost side (RHS) is increasing from 0 to 1. The benefit side decreases in α as its derivative is proportional to

$$\begin{aligned} \frac{\partial B_P^A}{\partial \alpha} &\propto \beta'_P(\alpha)\,(q\alpha + (1-q)\,\beta_P(\alpha)) - 2\beta_P(\alpha)\,(q + (1-q)\,\beta'_P(\alpha)) \\ &= -\left((1-q)\,\beta - q\alpha\right)\beta'_P(\alpha) - 2q\beta_P(\alpha) < 0 \end{aligned}$$

as

$$\alpha > \beta \quad \Longrightarrow \quad q\alpha < q\alpha\frac{F^{-1}(\alpha)}{F^{-1}(\beta)} = (1-q)\,\beta$$

For α approaching zero the benefit diverges as for any fixed N we have

$$\lim_{\alpha\to 0}\frac{1}{N}\frac{(1-q)\,\beta_P(\alpha)}{(q\alpha + (1-q)\,\beta_P(\alpha))^2} > \lim_{\alpha\to 0}\frac{1}{N}\frac{(1-q)}{\alpha}\frac{\beta_P}{\alpha} = \infty$$

because

$$\lim_{\alpha\to 0}\frac{\beta_P}{\alpha} = \lim_{\alpha\to 0}\frac{q}{1-q}\frac{F^{-1}(\alpha)}{F^{-1}(\beta_P)} > \frac{q}{1-q} > 0$$

For $\alpha = 1$ we have for all N above a certain value we have.

$$\frac{1}{N}\left(\frac{(1-q)\,\beta_P(1)}{(q + (1-q)\,\beta_P(1))^2}\right) < F^{-1}(1) = 1$$

Hence a unique solution $(\alpha_P, \beta_P(\alpha_P))$ exists for the PR equilibrium problem. $\square$

Proof or Proposition 7. .
The PR system gives immediately the underdog effect as F^{-1} is increasing

$$\begin{aligned} q\alpha F^{-1}(\alpha) &= (1-q)\,\beta F^{-1}(\beta) \\ q &< 1/2 \quad \iff \quad \alpha > \beta, \quad q\alpha < (1-q)\,\beta \end{aligned}$$

A larger population size N implies a lower expected turnout in the PR system. The marginal benefit side B_P^A decreases with N for all α while the cost side remains unchanged. Hence by the implicit function theorem as we increase N we have lower α which implies lower β and in turn lower turnout, formally

$$\begin{aligned} 0 &= \frac{d\left(B_P^A - F^{-1}\right)}{d\alpha}\frac{d\alpha}{dN} + \frac{d\left(B_P^A - F^{-1}\right)}{dN} \\ \frac{d\alpha}{dN} &= -\frac{\frac{dB_P^A}{dN}}{\frac{d\left(B_P^A - F^{-1}\right)}{d\alpha}} < 0 \implies \frac{d\beta}{dN} < 0 \implies \frac{dT}{dN} < 0 \end{aligned}$$

□

Proof of Proposition 8. For any $q \neq 1/2$ we need to show that for any N above a certain value we have for all $\alpha \in (0,1]$

$$\begin{aligned} B_M^A &< B_P^A \\ \frac{e^{-N\left(\sqrt{q\alpha}-\sqrt{(1-q)\beta_M}\right)^2}}{\sqrt{N}}\left(\frac{\sqrt{q\alpha}+\sqrt{(1-q)\,\beta_M}}{4\sqrt{\pi}\,(q\,(1-q)\,\alpha\beta_M)^{1/4}}\right)\frac{1}{\sqrt{q\alpha}} &< \frac{1}{N}\frac{(1-q)\,\beta_P}{(q\alpha+(1-q)\,\beta_P)^2} \end{aligned}$$

namely

$$e^{-N\left(\sqrt{q\alpha}-\sqrt{(1-q)\beta_M}\right)^2}\sqrt{N} < \frac{(1-q)\,\beta_P}{(q\alpha+(1-q)\,\beta_P)^2}\left(\frac{\sqrt{q\alpha}+\sqrt{(1-q)\,\beta_M}}{4\sqrt{\pi}\,(q\,(1-q)\,\alpha\beta_M)^{1/4}}\frac{1}{\sqrt{q\alpha}}\right)^{-1}$$

Since

$$\begin{aligned} \alpha &\in (0,1] \implies \beta_P \in (0,1], \quad \beta_M \in (0,1] \\ q &\neq 1/2 \implies \sqrt{q\alpha} \neq \sqrt{(1-q)\,\beta_M(\alpha)} \end{aligned}$$

the LHS converges to zero, whereas the RHS is a positive constant for all $\alpha \in (0,1]$. So for any $q \neq 1/2$ we need to show that for any N above a certain value

$$\alpha_M < \alpha_P$$

since in both MR and PR systems we have the symmetry property $\beta(q) = \alpha(1-q)$ the above also implies

$$\beta_M < \beta_P$$

For $q = 1/2$ we have $\alpha = \beta$ in both PR and MR systems. So for any N above a certain value and for all $\alpha \in (0, 1]$ we have that above a certain value of N

$$\begin{aligned} B_M^A &> B_P^A \\ \frac{1}{\sqrt{N}} \left(\frac{2\sqrt{q\alpha}}{4\sqrt{\pi}} \right) \frac{1}{q\alpha} &> \frac{1}{N} \left(\frac{q\alpha}{2\,(2q\alpha)^2} \right) \\ \sqrt{N} \left(\frac{1}{2\sqrt{\pi}\sqrt{q\alpha}} \right) &> \left(\frac{1}{8q\alpha} \right) \end{aligned}$$

as the RHS is a positive constant and the LHS increases to infinity.

As for underdog effects. Given that for MR we have

$$q\alpha_M \left(F^{-1}(\alpha_M) \right)^2 = (1-q)\, \beta_M \left(F^{-1}(\beta_M) \right)^2$$

and for PR we have

$$q\alpha_P \left(F^{-1}(\alpha_P) \right) = (1-q)\, \beta_P \left(F^{-1}(\beta_P) \right)$$

then, we have

$$1 < \frac{1-q}{q} = \frac{\alpha_P \left(F^{-1}(\alpha_P) \right)}{\beta_P \left(F^{-1}(\beta_P) \right)} = \frac{\alpha_M \left(F^{-1}(\alpha_M) \right)^2}{\beta_M \left(F^{-1}(\beta_M) \right)^2}$$

$q < 1/2$ implies that we have underdog effects for the two models

$$\alpha_P > \beta_P, \quad \alpha_M > \beta_M$$

and also implies that the leading party obtains the majority

$$q\alpha_P < (1-q)\, \beta_P, \quad q\alpha_M < (1-q)\, \beta_M$$

Finally, since

$$F^{-1}(\alpha_M) > F^{-1}(\beta_M)$$

and F^{-1} is increasing, it is always true that

$$\frac{\alpha_P F^{-1}(\alpha_P)}{\beta_P F^{-1}(\beta_P)} = \frac{\alpha_M \left(F^{-1}(\alpha_M) \right)^2}{\beta_M \left(F^{-1}(\beta_M) \right)^2} > \frac{\alpha_M F^{-1}(\alpha_M)}{\beta_M F^{-1}(\beta_M)}$$

□

Proof of Lemma 9. Express the following series by differentiating and integrating the summands and inverting the series and integral operators

$$
\begin{aligned}
\sum_{b=0}^{\infty} \frac{B^b}{b!} \frac{a}{a+b+c} &= \frac{a}{B^{a+c}} \sum_{b=0}^{\infty} \int_0^B \frac{d}{dr} \left(\frac{1}{b!} \frac{r^{a+b+c}}{a+b+c} \right) dr \\
&= \frac{a}{B^{a+c}} \int_0^B \sum_{b=0}^{\infty} \left(\frac{1}{b!} r^{a+b+c-1} \right) dr \\
&= \begin{cases} \frac{a}{B^{a+c}} \int_0^B r^{a+c-1} e^r dr & \text{for } a \geq 1 \\ 1/3 & \text{for } a = c = 0 \end{cases}
\end{aligned}
$$

and

$$
\sum_{b=0}^{\infty} \frac{B^b}{b!} \frac{a+1}{a+b+1} = \frac{a+1}{B^{a+c+1}} \int_0^B r^{a+c} e^r dr
$$

We compute the marginal benefit for party A by inverting the series and integral operators again over the series over a.

$$
\begin{aligned}
B_P^A &= e^{-(A+B+C)} \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \left(\begin{array}{c} \sum_{a=0}^{\infty} \frac{A^a}{a!} \left(\frac{a+1}{B^{a+c+1}} \int_0^B r^{a+c} e^r dr \right) \\ - \sum_{a=1}^{\infty} \frac{A^a}{a!} \left(\frac{a}{B^{a+c}} \int_0^B r^{a+c-1} e^r dr \right) \end{array} \right) - \frac{1}{3} \right) \\
&= e^{-(A+B+C)} \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \left(\begin{array}{c} \int_0^B \frac{r^c}{B^{c+1}} \left(\sum_{a=0}^{\infty} \frac{(Ar/B)^a}{(a-1)!} + \sum_{a=0}^{\infty} \frac{(Ar/B)^a}{a!} \right) e^r dr \\ - \int_0^B \frac{r^{c-1}}{B^c} \left(\sum_{a=1}^{\infty} \frac{(Ar/B)^a}{(a-1)!} \right) e^r dr \end{array} \right) - \frac{1}{3} \right) \\
&= e^{-(A+B+C)} \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \left(\begin{array}{c} \int_0^B \frac{r^c}{B^{c+1}} \left((Ar/B)\, e^{(Ar/B)} + e^{(Ar/B)} \right) e^r dr \\ - \int_0^B \frac{r^{c-1}}{B^c} \left((Ar/B)\, e^{(Ar/B)} \right) e^r dr \end{array} \right) - \frac{1}{3} \right)
\end{aligned}
$$

Inverting the series and integral operators again over the series over c.

$$
\begin{aligned}
B_P^A &= e^{-(A+B+C)} \left(\begin{array}{c} \int_0^B \left((Ar/B)\, e^{(Ar/B)} + e^{(Ar/B)} \right) \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \frac{r^c}{B^{c+1}} \right) e^r dr \\ - \int_0^B \left((Ar/B)\, e^{(Ar/B)} \right) \left(\sum_{c=0}^{\infty} \frac{C^c}{c!} \frac{r^{c-1}}{B^c} \right) e^r dr - \frac{1}{3} \end{array} \right) \\
&= e^{-(A+B+C)} \left(\begin{array}{c} \int_0^B \left((Ar/B)\, e^{(Ar/B)} + e^{(Ar/B)} \right) \frac{e^{rC/B}}{B} e^r dr \\ - \int_0^B \left((Ar/B)\, e^{(Ar/B)} \right) \frac{e^{rC/B}}{r} e^r dr - \frac{1}{3} \end{array} \right) \\
&= e^{-(A+B+C)} \left(\int_0^B \left(\left(\frac{A}{B} r e^{\frac{A+B+C}{B} r} + e^{\frac{A+B+C}{B} r} \right) \frac{1}{B} - \frac{A}{B} e^{\frac{A+B+C}{B} r} \right) dr - \frac{1}{3} \right)
\end{aligned}
$$

Computing the integral and simplifying, we have

$$
\begin{aligned}
B_P^A &= e^{-(A+B+C)} \left(\left(\begin{array}{c} \left(AB\left(\frac{1-e^{A+B+C}}{(A+B+C)^2}+\frac{e^{A+B+C}}{A+B+C}\right)+\left(B\frac{e^{A+B+C}-1}{A+B+C}\right)\right)\frac{1}{B} \\ -\frac{A}{B}\left(B\frac{e^{A+B+C}-1}{A+B+C}\right) \end{array} \right) - \frac{1}{3} \right) \\
&= e^{-(A+B+C)} \left(A\frac{1-e^{A+B+C}}{(A+B+C)^2} + \frac{e^{A+B+C}-1}{A+B+C} + \frac{A}{A+B+C} - \frac{1}{3} \right) \\
&= \frac{B+C}{(A+B+C)^2}\left(1-e^{-(A+B+C)}\right) + \left(\frac{A}{A+B+C} - \frac{1}{3}\right) e^{-(A+B+C)} \\
&= \left(1 - \frac{A}{A+B+C}\right)\frac{1-e^{-(A+B+C)}}{A+B+C} + \left(\frac{A}{A+B+C} - \frac{1}{3}\right) e^{-(A+B+C)}
\end{aligned}
$$

□

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