Market distortions and local indeterminacy: a general approach

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Abstract

We provide a general methodology to study the role of market distortions on local indeterminacy and bifurcations. We extend the well-known Woodford (1986) model to account for market distortions, introducing general specifications for three crucial functions: the real interest rate, the real wage and the workers' offer curve. The elasticities of these three functions play a key role on local dynamics and allow us to identify which types of distortions are the most powerful for indeterminacy.

Most of the specific market imperfections considered in the related literature are particular cases of our general framework. Comparing them we obtain several equivalence results in terms of indeterminacy mechanisms. We also provide examples of distortions that illustrate new results. Furthermore we show that, for an elasticity of substitution between inputs around unity, indeterminacy requires a minimal degree of distortions. However, the degree of labor market distortions compatible with that requirement is empirically plausible.

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1 Introduction

In this paper we develop a methodology to study and fully characterize the role of market distortions on the occurrence of local indeterminacy and bifurcations. Several papers have been studying the effects of certain specific market distortions (linked to externalities, imperfectly competitive markets, or government intervention) on local dynamics,[1] but a systematic analysis within a general unified framework, able to compare the importance of different types of distortions as a route to indeterminacy and bifurcations, is still missing. In order to do that we introduce a general framework, able to account for market distortions without specifying a priori their source, and highlight the main channels through which indeterminacy occurs.

Although our methodology can be applied to any dynamic general equilibrium model, the dynamic framework considered in this paper is based on the perfectly competitive one sector model of a segmented asset economy of Woodford (1986) and Grandmont et al. (1998).[2] Market distortions play a role on the local stability properties of the steady state because they modify the elasticities of three crucial functions that characterize our two dimensional equilibrium dynamic system: the real interest rate, the real wage or equivalently effective consumption per unit of labor, and the generalized offer curve. We introduce general specifications for these elasticities that allow us to recover most of the distortions on product, capital and labor markets, and admit perfect competition as a particular case.

Focusing on not too weak values of the elasticity of capital-labor substitution,[3] we show that, in contrast to the perfectly competitive economy,[4] when there are market distortions, indeterminacy and bifurcations may occur in the presence of sufficiently high capital-labor substitution and labor supply elasticities. However, in some cases, indeterminacy is ruled out if the individual labor supply elasticity becomes arbitrarily large, implying that, by imposing an infinitely elastic labor supply, one may obtain a wrong idea of the dynamic implications of some distortions. We find that distortions affecting the real interest rate do not play a major role. On the contray, even (arbitrarily) small distortions on the offer curve and/or effective consumption

[1]See the survey by Benhabib and Farmer (1999) and the bibliographic references in Section 4.

[2]This is a suitable framework for our purpose, since several papers have introduced specific market distortons on product and factor markets in this model. These papers provide examples to apply our general methodology (see Section 4).

[3]Weak values of this elasticity are not empirically relevant (Hamermesh (1993), Duffy and Papageorgiou (2000)).

[4]Indeterminacy only occurs under perfect competition (Woodford (1986), Grandmont et al. (1998)) for a weak capital-labor substitution.

promote the occurrence of indeterminacy. However, indeterminacy can only prevail for values of the elasticity of capital-labor substitution around one (those considered empirically plausible), under a minimal level of distortions in the offer curve and/or in effective consumption. Furthermore, distortions modifying the offer curve affect significantly the emergence of indeterminacy, leading to new results.

To clarify these findings, we apply our general method to examples of specific distortions on the product, capital and labor markets, focusing on two types of results. On one hand, we obtain several equivalence results in terms of local dynamics and indeterminacy mechanisms. We find that labor and consumption taxation with a balanced budget are equivalent to consumption externalities, sharing the same indeterminacy mechanism. Product market imperfections (due to mark-up variability and taste for variety) can be seen as particular cases of the framework with positive externalities in production, and unemployment benefits with efficiency wages can be recovered as a particular case of an economy where the desutility of labor is negatively affected by labor externalities. On the other hand, we also discuss the degrees of specific market distortions required for the occurrence of indeterminacy under empirically plausible values (i.e. around one) of the elasticity of capital-labor substitution. We confirm, focusing on capital income taxation, that distortions on the capital market do not, per se, promote indeterminacy. Under output market distortions, indeterminacy may emerge, but requires parameters configurations at odds with empirical evidence, i.e., too strong positive externalities in production or high markups. On the contrary, under labor market distortions (unions, efficiency wages, unemployment benefits, externalities in preferences), indeterminacy and bifurcations emerge for empirically plausible distortions. Hence, labor market distortions are the most relevant for indeterminacy.

The rest of the paper is organized as follows. We present our general framework in Section 2, study the role of distortions on local dynamics in Section 3, and apply our results to examples with specific market distortions in Section 4. Section 5 provides concluding remarks. Proofs and technical details are provided in the Appendix.

2 The model

Our framework extends the perfectly competitive Woodford model to take into account market imperfections. To ease the presentation we begin with a brief exposition of this model.

According to the perfectly competitive economy studied by Woodford

(1986) and Grandmont et al. (1998), in each period $t \in N^*$, a final good is produced under a constant returns to scale technology $AF(K_{t-1}, L_t)$, where $A > 0$ is a scaling parameter, $F(K, L)$ is a strictly increasing function, concave and homogeneous of degree one in capital, $K > 0$, and labor, $L > 0$. From profit maximization, the real interest rate ρ_t and the real wage ω_t are respectively equal to the marginal productivities of capital and labor, i.e. $\rho_t = AF_K(K_{t-1}, L_t) \equiv A\rho(K_{t-1}/L_t)$ and $\omega_t = AF_L(K_{t-1}, L_t) \equiv A\omega(K_{t-1}/L_t)$. There are two types of infinitely-lived consumers, workers and capitalists. Both can save through two assets, money and productive capital. However, capitalits are less impatient than workers and do not supply labor, whereas workers face a finance constraint which prevents them from borrowing against their wage earnings. Focusing on equilibria where the finance constraint is binding and capital is the asset with the greatest return, it follows that only workers hold money (they save all their wage income in money), and capitalists hold the entire stock of capital. As in Woodford (1986), the behaviour of the representative worker can be summarized by the maximization of $U\left(C_{t+1}^w/B\right) - V(L_t)$ subject to the budget constraint $P_{t+1}C_{t+1} = w_t L_t$, where P_t is the price of the final good and w_t the nominal wage at period t, $C_{t+1}^w \geq 0$ the worker's consumption at period $t+1$, $B > 0$ a scaling parameter, $V(L)$ the desutility of labor in $L \in [0, L^*]$ and $U(C^w/B)$ the utility of consumption.[5] The solution of this problem is given by the intertemporal trade-off between future consumption and leisure:

$$\omega_{t+1}L_{t+1}/B = \gamma(L_t) \tag{1}$$

where $\gamma(L)$ is the usual offer curve and $C_{t+1}^w = \omega_{t+1}L_{t+1}$ at the monetary equilibrium, with a fixed constant amount of money in the economy.

The representative capitalist maximizes the log-linear lifetime utility function $\sum_{t=1}^{\infty} \beta^t \ln C_t^c$ subject to the budget constraint $C_t^c + K_t = (1 - \delta + r_t/P_t)K_{t-1}$, where C_t^c represents his consumption at period t, $\beta \in (0,1)$ his subjective discount factor, r_t the nominal interest rate and $\delta \in (0,1)$ the depreciation rate of capital. Solving the capitalist's problem we obtain the capital accumulation equation

$$K_t = \beta\left[1 - \delta + \rho_t\right] K_{t-1} \tag{2}$$

A perfectly competitive intertemporal equilibrium is a sequence (K_{t-1}, L_t)

[5]It is assumed that $U\left(C_{t+1}^w/B\right)$ is a continuous function of $C_{t+1}^w \geq 0$, and C^r, with r high enough, $U' > 0, U'' \leq 0$ for $C_{t+1}^w > 0$, and $-xU''(x)/U'(x) < 1$. Also, $V(l)$ is a continuous function for $[0, L^*]$, and C^r, with r high enough, $V' > 0, V'' \geq 0$ for $(0, L^*)$. We also assume that $lim_{L \to L^*} V'(L) = +\infty$, with L^* (the worker's endowment) possibly infinite.

$\in \mathbb{R}^2_{++}$, $t = 1, 2, ..., \infty$, given $K_0 > 0$, satisfying (1) and (2), where $\omega_t = A\omega(K_{t-1}/L_t)$ and $\rho_t = A\rho(K_{t-1}/L_t)$.[6]

We now present our general framework with market distortions, explaining in a second step the main differences with respect to the perfectly competitive case. We propose a more general equilibrium dynamic system, given by (3)-(4) in Definition 1 below, where ϱ_t represents the real interest rate relevant to capitalists' decisions, Γ_t a generalized offer curve, and $\Omega_t L_t$ effective consumption. In what follows, we denote by $\varepsilon_{X,y}$ the elasticity, evaluated at the steady state, of the function $X = \{\varrho, \Omega, \Gamma\}$ with respect to the argument $y = \{K, L\}$, while $\varepsilon_\gamma - 1 \geqslant 0$ is the inverse of the elasticity of labor supply of the representative worker with respect to labor, $s \in (0, 1)$ the elasticity of the production function with respect to capital, and $\sigma > 0$ is the elasticity of capital-labor substitution of the representative firm, all evaluated at the private level and at the steady state.[7]

Definition 1 *A perfect foresight intertemporal equilibrium of the economy with market distortions is a sequence $(K_{t-1}, L_t) \in \mathbb{R}^2_{++}$, $t = 1, 2, ..., \infty$, that for a given $K_0 > 0$ satisfies:*

$$K_t = \beta \left[1 - \delta + \varrho_t\right] K_{t-1} \tag{3}$$

$$(1/B)\Omega_{t+1}L_{t+1} = \Gamma_t \tag{4}$$

where $\varrho_t \equiv A\varrho(K_{t-1}, L_t)$, $\Omega_t \equiv A\Omega(K_{t-1}, L_t)$ and $\Gamma_t \equiv \Gamma(K_{t-1}, L_t)$. The functions $\varrho(K, L)$, $\Omega(K, L)$ and $\Gamma(K, L)$ are positively valued and differentiable as many times as needed for $(K, L) \in \mathbb{R}^2_{++}$, such that

$$\begin{aligned} \varepsilon_{\varrho,K} &= \alpha_{K,K} + \frac{\beta_{K,K}}{\sigma} - \frac{1-s}{\sigma}\ , \ \varepsilon_{\varrho,L} = \alpha_{K,L} + \frac{\beta_{K,L}}{\sigma} + \frac{1-s}{\sigma} \\ \varepsilon_{\Omega,K} &= \alpha_{L,K} + \frac{\beta_{L,K}}{\sigma} + \frac{s}{\sigma}\ , \ \varepsilon_{\Omega,L} = \alpha_{L,L} + \frac{\beta_{L,L}}{\sigma} - \frac{s}{\sigma} \\ \varepsilon_{\Gamma,K} &= \alpha_{\Gamma,K} + \frac{\beta_{\Gamma,K}}{\sigma}\ , \ \varepsilon_{\Gamma,L} = \alpha_{\Gamma,L} + \frac{\beta_{\Gamma,L}}{\sigma} + \varepsilon_\gamma, \end{aligned} \tag{5}$$

where $\alpha_{i,j} \in \mathbb{R}$ and $\beta_{i,j} \in \mathbb{R}$, for $i = K,\ L,\ \Gamma$ and $j = K, L$, are parameters independent of ε_γ and σ

As under perfect competition, the dynamics of the economy with market distortions are governed by a two dimensional system in capital and labor, where the first equation represents capital accumulation and the second one

[6] See Grandmont et al. (1998) and Woodford (1986) for more details.

[7] We consider the normalized steady state $(K, L) = (1, 1)$ of the dynamic system (3)-(4), whose existence is shown in Proposition 2 of Appendix 6.1.

the intertemporal choice of workers. However, now we assume that ϱ_t, Ω_t and Γ_t are given by general functions of K_{t-1} and L_t, without choosing a particular specification for them, so that they encompass a large class of specific market distortions. Of course, the perfectly competitive case is recovered from Definition 1 for $\varrho(K, L) = F_K(K, L)$, $\Omega(K, L) = F_L(K, L)$ and $\Gamma(K, L) = \gamma(L)$. However, with market distortions, ϱ_t may not coincide with the perfectly competitive marginal productivity of capital, $\Omega_t L_t$ may not coincide with the perfectly competitive wage bill and Γ_t may differ from the private offer curve $\gamma(L_t)$. For example, in the cases of productive externalities, imperfect competition in the product market or with consumption, labor or capital taxation, the real interest rate and/or the real wage relevant to consumers' decisions are no longer equal to the marginal productivities of capital and labor at the firm level. Also in the case of consumption or public spending externalities on preferences the relevant intertemporal choice of workers becomes a trade-off between future effective consumption[8], $\Omega_{t+1}L_{t+1,}$ (that no longer coincides with the wage bill) and leisure. Moreover, in the presence of some labor market imperfections, such as efficiency wages or unions, or with leisure externalities, the private offer curve derived for the perfectly competitive economy is no longer valid at the social level, where the relevant concept is the generalized offer curve Γ. See the examples provided in Section 4.

Market distortions modify the elasticities of these three functions, with respect to the perfectly competitive case. When $\alpha_{i,j} = \beta_{i,j} = 0$, we recover the elasticities under perfect competition[9]. Hence, in each equality, $\alpha_{i,j} + \beta_{i,j}/\sigma \neq 0$ represent market distortions, which add two new components to the different elasticities: $\alpha_{i,j}$ which provides a measure of the importance of market distortions when inputs are high substitutes in production (σ high), and $\beta_{i,j}$, which become more relevant when inputs are weak substitutes in production (σ low).

In the rest of the paper we consider that $|\beta_{i,j}| < s$, for all $i = K, L, \Gamma$ and $j = K, L$. Since empirical works show that market distortions are not too big, this is a plausible assumption covering the most interesting cases presented in the literature. Empirical studies also show that the wage bill is increasing in labor. Without market distortions this means that consumption is increasing in labor. We extend this assumption to our economy with market distortions, so that effective consumption (ΩL) is increasing in labor, i.e. $1 + \epsilon_{\Omega,L} > 0$, implying, from Definition 1, $\alpha_{LL} > -1$ and $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$. In accordance

[8]By effective consumption we mean the argument of the utility for consumption, which in the presence of consumption or public spending externalities on preferences will also include them.

[9]See Grandmont et al. (1998).

with empirical studies we also assume that capital income is increasing with capital, i.e. $1+\theta\epsilon_{\varrho,K}>0$ where $\theta\equiv 1-\beta(1-\delta)\in(0,1)$, which, under Definition 1, implies $\alpha_{K,K}>-1/\theta$ and $\sigma>\frac{\theta(1-s-\beta_{KK})}{1+\theta\alpha_{KK}}$. Finally, we consider that $s<1/2$ and $\theta<s/(1-s)$, as usually done in Woodford economies,[10] and we further assume that $\frac{s-\beta_{LL}}{1+\alpha_{LL}}>\frac{\theta(1-s-\beta_{KK})}{1+\theta\alpha_{KK}}$.[11] We summarize all the conditions discussed above in the following Assumption:

Assumption 1

1. $0<s<1/2$ *and* $0<\theta(1-s)<s$.
2. $|\beta_{i,j}|<s$, *for all* $i=K,L,\Gamma$ *and* $j=K,L$;
   $\alpha_{LL}>-1,\ \ \alpha_{K,K}>-1/\theta$ *and* $\frac{s-\beta_{LL}}{1+\alpha_{LL}}>\frac{\theta(1-s-\beta_{KK})}{1+\theta\alpha_{KK}}$.
3. $\sigma>\frac{s-\beta_{LL}}{1+\alpha_{LL}}$.

Assumption 1.3 implies that we only focus on not too weak values of the elasticity of capital-labor substitution.[12] It collapses into $\sigma>s$ in the absence of distortions, a case where, as shown in Grandmont et al. (1998), indeterminacy and bifurcations are not possible. Hence, the occurrence of indeterminacy and bifurcations in our framework is due to the existence of market distortions, mainly through their effects on $\alpha_{i,j}$, which are more relevant than $\beta_{i,j}$ when inputs are not weak substitutes in production.

By loglinearizing the system (3)-(4) around the normalized steady state, we obtain the local dynamics for $\widehat{K}_t=(K_t-K)/K$ and $\widehat{L}_{t+1}=(L_{t+1}-L)/L$ given by following equations:

$$\begin{bmatrix}\widehat{K}_t\\ \widehat{L}_{t+1}\end{bmatrix}=\begin{bmatrix}(1+\theta\varepsilon_{\varrho,K}) & \theta\varepsilon_{\varrho,L}\\ \frac{\epsilon_{\Gamma,K}-\varepsilon_{\Omega,K}(1+\theta\varepsilon_{\varrho,K})}{1+\varepsilon_{\Omega,L}} & \frac{\varepsilon_{\Gamma,L}-\theta\varepsilon_{\Omega,K}\varepsilon_{\varrho,L}}{1+\varepsilon_{\Omega,L}}\end{bmatrix}\begin{bmatrix}\widehat{K}_{t-1}\\ \widehat{L}_t\end{bmatrix}\equiv[J]\begin{bmatrix}\frac{K_{t-1}-K}{K}\\ \frac{L_t-L}{L}\end{bmatrix}\tag{6}$$

Market distortions influence the local dynamics of the model, relatively to the perfectly competititive case, by modifying the elasticities $\varepsilon_{\Omega,i}$, $\varepsilon_{\varrho,i}$ and $\varepsilon_{\Gamma,i}$. Since θ is small and $\varepsilon_{\varrho,i}$ appears multiplied by θ, distortions on the

[10] See, for instance, Grandmont and al. (1998), Cazzavillan et. al. (1998), Barinci and Chéron (2001), Lloyd-Braga and Modesto (2007), Dufourt et al. (2008)

[11] This is not restrictive, since empirical values of θ are rather small (around 0.0123 under most monthly parameterizations). Furthermore, it becomes $s>\theta(1-s)$ in the absence of market imperfections.

[12] It also covers the most empirically relevant situations. For instance Hamermesh (1993) and Duffy and Papageorgiou (2000) find values in accordance with σ greater than 0.4.

ϱ function do not play a major role. However, distortions affecting the Γ and/or the Ω function can significantly affect the dynamic behaviour of our system. Indeed, under Assumption 1, in the perfectly competitive case we have $\varepsilon_{\Gamma,L} > 1$ and $0 < 1 + \varepsilon_{\Omega,L} < 1$, while with distortions, by (5), $\varepsilon_{\Gamma,L}$ may become lower than 1 ($\alpha_{\Gamma,L} < 0$) and $1 + \varepsilon_{\Omega,L}$ greater than 1 ($\alpha_{L,L} > 0$). Due to this, as we shall see in next section, indeterminacy and bifurcations occur when capital and labor are not weak substitutes, even if there are only arbitrarily small distortions in the Γ and/or the Ω function. Finally, a major and novel difference with respect to the perfectly competitive case is that in our framework, if $\alpha_{\Gamma,L}$ is sufficiently negative, $\varepsilon_{\Gamma,L}$, may become negative. This leads to new parameters' configurations under which indeterminacy occurs.[13]

3 The role of market distortions on local dynamics

In order to study the role of market distortions on local dynamics, we first obtain the trace, T, and the determinant, D, of the Jacobian matrix, J, of system (6). We then analyze the occurrence of indeterminacy and bifurcations by studying how T and D evolve in the space (T, D) as some relevant parameters of the model are made to vary continuously in their admissible range, according to the geometrical method developed in Grandmont et al. (1998). Below we only discuss some necessary conditions required for indeterminacy and bifurcations, and we summarize our results in Proposition 1 and Tables 1 and 2. Proofs, technical details, and a complete use of the geometrical method are provided in the Appendix.

Note that T and D correspond, respectively, to the sum and product of the two roots of the associated characteristic polynomial $P(\lambda) \equiv \lambda^2 - \lambda T + D$. Hence, the local dynamic properties of the model depend on the values taken by T and D. To locate these values in the plane (T, D), three lines are relevant (see Figure ??). On the line (AB), one eigenvalue is equal to -1, i.e. $P(-1) \equiv 1 + T + D = 0$. On the line (AC), one eigenvalue is equal to 1, i.e. $P(1) \equiv 1 - T + D = 0$. On the segment $[BC]$, the two eigenvalues are complex conjugates with a unit modulus, i.e. $D = 1$ and $|T| < 2$. It can be deduced that the steady state is a sink (indeterminate)[14] when $D < 1$ and $|T| < 1 + D$, i.e., (T, D) is inside the triangle (ABC). It is a saddle-point when

[13] Although not yet explored in the literature, specific distortions leading to this situation exist. See the last example of section 4.4.

[14] When the steady state is locally indeterminate there is, for each given value K_0 close to the steady state, a continuum of deterministic equilibrium trajectories $\{K_{t-1}, L_t\}_{t=1,2,\ldots,\infty}$, all converging to the steady state.

$|1 + D| < |T|$. Otherwise, it is a source (locally unstable). Also, considering, for instance, that ε_γ is running the interval $[1, +\infty)$, a transcritical bifurcation generically occurs when (T, D) crosses the line (AC), i.e. when ε_γ crosses the critical value ε_{γ_T}.[15] When (T, D) crosses the line (AB), ε_γ crossing the critical value ε_{γ_F}, a flip bifurcation generically occurs. When (T, D) crosses the segment $[BC]$ in its interior, ε_γ crossing the critical value ε_{γ_H}, a Hopf bifurcation generically occurs.[16]

In our framework, using (5), the trace and the determinant of the Jacobian matrix, J given in (6), can be written in terms of the parameters of the model:

$$\begin{aligned} T &= T_0(\sigma)(\varepsilon_\gamma - 1) + T_1(\sigma) && (7) \\ D &= D_0(\sigma)(\varepsilon_\gamma - 1) + D_1(\sigma), && (8) \end{aligned}$$

where $T_h(\sigma)$ and $D_h(\sigma)$, $h = 1, 2$, are also functions of α_{ij}, β_{ij}, θ and s (see (11) and (12)).

Under Assumption 1, $D_0(\sigma) > 0$, hence $D \geq D_1(\sigma)$ for $\varepsilon_\gamma \geq 1$. When $D_1'(\sigma) < 0$, we further have $D_1(\sigma) > D_1(+\infty)$. Therefore, the necessary condition $D < 1$ for indeterminacy requires that $D_1(+\infty) < 1$ when $D_1'(\sigma) < 0$. In the rest of the paper we consider that the condition $D_1(+\infty) < 1$ is always satisfied. We further assume that $D_1(+\infty) > -1$. Using (15), we summarize these conditions as follows:

Assumption 2

1. $D_1(+\infty) < 1$, *i.e.*, $\alpha_{L,L} - \alpha_{\Gamma,L} > \theta[\alpha_{K,K}(1 + \alpha_{\Gamma,L}) - \alpha_{\Gamma,K}\alpha_{K,L}]$.

2. $D_1(+\infty) > -1$, *i.e.*, $\theta[\alpha_{K,K}(1+\alpha_{\Gamma,L})-\alpha_{\Gamma,K}\alpha_{K,L}] > -(2 + \alpha_{L,L} + \alpha_{\Gamma,L})$.

[15] A saddle node or a pitchfork bifurcation may be also possible. The case of a saddle node bifurcation, in which the steady state under analysis disappears, is ruled out, since we apply our analysis to $(K, L) = (1, 1)$ whose existence is persistent, under the usual scaling procedure. We also assume that pitchfork bifurcations are ruled out, as a mere exposition device. Notice that, several works that have studied the existence of multiple steady states in economies with constant elasticities ϵ_γ and σ (eg. Cazzavillan et al. (1998) and Kuhry (2001)) found at most two steady states, which rules out the case of a pitchfork bifurcation.

[16] The expressions of ε_{γ_T}, ε_{γ_F} and ε_{γ_H} are given in Appendix 6.5. Moreover, when the steady state undergoes a Hopf or a flip bifurcation there are, for some values of ε_γ close to, respectively ε_{γ_H} or ε_{γ_F}, deterministic equilibrium trajectories exhibiting recurrent bounded fluctuations. When a transcritical bifurcation occurs, two nearby steady states exchange stability properties. Note also that bifurcations are quite relevant in explaining persistency of business fluctuations, since they appear when at least one eigenvalue crosses the unit circle.

The second inequality is obviously satisfied without distortions, and, therefore, is also verified for values of $\alpha_{i,j}$ not arbitrarily large. Since we are not interested in unrealistically strong market distortions, this seems to be a reasonable assumption. The first condition, necessary for indeterminacy when $D_1'(\sigma) < 0$, can only be met with distortions. Indeed, under perfect competition $D_1'(\sigma) < 0$ but $D_1(+\infty) = 1$, which rules out indeterminacy.

We now present our results, considering separately two cases: $D_1'(\sigma) < 0$ and $D_1'(\sigma) > 0$, that emerge for different values of the distortion parameters. See (18). Within each case we use a classification based on the values taken by $S_1 = D_1'(\sigma)/T_1'(\sigma)$, which depends on the parameters $\alpha_{i,j}$, $\beta_{i,j}$, θ and s, and whose expression is given in (16) in the Appendix. We obtain six configurations. $D_1'(\sigma) < 0$ in configuration (i), (ii), (iii) and (iv), presented in Table 1, whereas $D_1'(\sigma) > 0$ in configurations (v) and (vi) in Table 2.

When distortions only appear on the ϱ and Ω functions, we always obtain $D_1'(\sigma) < 0$. See (19). Hence, configurations (v) and (vi), where $D_1'(\sigma) > 0$, can only emerge in the presence of distortions on the Γ function ($\alpha_{\Gamma,i} \neq 0$ and/or $\beta_{\Gamma,i} \neq 0$).[17] Moreover, since under θ small, distortions in ρ do not to play a major role on local dynamics and are not required to obtain $D_1'(\sigma) > 0$, we only consider distortions on the Γ and Ω functions when analyzing this case. We further notice that when $D_1'(\sigma) > 0$ and $\alpha_{K,i} = \beta_{K,i} = 0$, the necessary condition for indeterminacy $D > T - 1$ requires $S_1 \in (0,1)$. See (22) in Appendix 6.3. This explains why we do not consider configurations where $S_1 \notin (0,1)$ when $D_1'(\sigma) > 0$. Finally, it is useful to notice that in the limit case of perfect competition ($\alpha_{i,j} = \beta_{i,j} = 0$) and under Assumption 1, we have $S_1 = 1$ and $D_1'(\sigma) < 0$, using (16) and (19). Therefore, configurations (i) and (ii) of Table 1 correspond to the smaller departure from the case without distortions. Proposition 1 below summarizes our results on local stability properties and bifurcations:

Proposition 1 *Let $(K, L) = (1, 1)$ be the normalized steady state of the dynamic system (3)-(4), as stated in Proposition 2. Consider that σ takes values in intervals specified by referring to the critical values σ_T, σ_F, σ_{H_1}, σ_{H_2}, σ_{H_3} and σ^{S_2} defined in Appendix 6.6. Take $\varepsilon_\gamma \in [1, \infty)$ as the bifurcation parameter with Hopf, flip and transcritical bifurcation values ε_{γ_H} ε_{γ_F} and ε_{γ_T}, respectively given in (23),(24) and (25).*[18]

(a) Consider that α_{KK} takes either nonpositive or positive values. Then,

[17] This is the case where D can take negative values. See figures 6 and 7.

[18] Assumptions 3 and 4 are satisfied in all the examples considered in the related literature, and simplify considerably our analysis. Assumptions 5, 6 and 7 are merely introduced as an exposition device. See Appendices 6.2 and 6.3. The critical values of S_B and S_D, used to define some configurations, are given in Appendix 6.6.

under Assumptions 1-4, and imposing also Assumptions 5 and 6 in the case of configurations (ii), (iii) and (iv), the nature of the steady state, whether a saddle, a sink or a source, depends upon the values of the parameters α_{KK}, σ *and* ε_γ *belonging to the intervals indicated in Table 1.*[19]

(b) Assume that $\alpha_{Ki} = \beta_{Ki} = 0$ *and consider that* $(\alpha_{\Gamma,K}-\alpha_{L,K})$ *takes either nonpositive or positive values. Then, under Assumptions 1-4, and imposing also Assumption 7 in the case of configurations (vi), the nature of the steady state, whether a saddle, a sink or a source, depends upon the values of the parameters* $(\alpha_{\Gamma,K}-\alpha_{L,K})$, σ *and* ε_γ *belonging to the intervals indicated in Table 2.*

Also, whenever the critical value ε_{γ_H} *(resp.* ε_{γ_F} *or* ε_{γ_T}*) appears in some row of Tables 1 and 2, a Hopf bifurcation (resp. a flip or transcritical bifurcation) generically occurs as* ε_γ *crosses the corresponding value.*

Proof. See Appendix 6.3.

3.1 Discussion of the results

By inspection of Tables 1 and 2, we see that, when capital and labor are sufficiently substitutable in production, indeterminacy and bifurcations occur in the presence of market distortions.[20]

Indeterminacy requires a critical lower bound on the elasticity of substitution between capital and labor (σ), which may be different across configurations. In Table 1, where $D_1'(\sigma) < 0$, this critical lower bound is higher or equal to σ_{H_1}, because $\sigma > \sigma_{H_1}$ is a necessary condition for indeterminacy.[21] In Table 2, where $D_1'(\sigma) > 0$, although $\sigma > \sigma_{H_1}$ is no longer a necessary condition for indeterminacy, indeterminacy also requires a lower bound for σ.

Indeterminacy also requires a critical upper bound (either $\epsilon_{\gamma H}$ or $\epsilon_{\gamma T}$) on ϵ_γ, depending on the configuration considered. However, in configurations (i), (iv), (v) and (vi) indeterminacy may be ruled out if ϵ_γ is small and sufficiently close to 1. Therefore, imposing an infinitely elastic labor supply at

[19] In Table 1 the lines with a * disappear if σ_{H_2} does not exist and the upper limit of the preceding lines becomes ∞.

[20] In Grandmont et al. (1998), the consequences of indeterminacy and bifurcations for the existence and properties of equilibria, in particular those exhibiting deterministic and stochastic expectations-driven cycles, are explored.

[21] Indeterminacy requires $D < 1$. Since D is increasing with ε_γ, $D_1(\sigma) < 1$ is needed, which is equivalent to $\sigma > \sigma_{H_1}$ when $D_1'(\sigma) < 0$ as in Table 1.

			ε_γ		
Config.		σ	*Saddle*	*Sink*	*Source*
(i)	1) $\alpha_{K,K} \leq 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \infty\right)$	$[1, \infty)$	-	-
$D_1'(\sigma) < 0$	2) $\alpha_{K,K} > 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_T\right]$	$[1, \infty)$	-	-
$S_1 \in (0, 1)$		(σ_T, σ_{H_2})	$[1, \varepsilon_{\gamma_T})$	-	$(\varepsilon_{\gamma_T}, \infty)$
		(σ_{H_2}, ∞)	$[1, \varepsilon_{\gamma_T})$	$(\varepsilon_{\gamma_T}, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \infty)$
(ii)	1) $\alpha_{K,K} \leq 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_{H_1}\right]$	$(\varepsilon_{\gamma_T}, \infty)$	-	$[1, \varepsilon_{\gamma_T})$
$D_1'(\sigma) < 0$		$(\sigma_{H_1}, \sigma_{H_2})$	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T})$
$\lvert S_1 \rvert > 1$	*	(σ_{H_2}, ∞)	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_T})$	-
	2) $\alpha_{K,K} > 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_{H_1}\right]$	$(\varepsilon_{\gamma_T}, \infty)$	-	$[1, \varepsilon_{\gamma_T})$
		(σ_{H_1}, σ_T)	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T})$
		(σ_T, ∞)	-	$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \infty)$
(iii)	1) $\alpha_{K,K} \leq 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_F\right)$	$[1, \varepsilon_{\gamma_F})$	-	$(\varepsilon_{\gamma_F}, \varepsilon_{\gamma_T})$
$D_1'(\sigma) < 0$			*and* $(\varepsilon_{\gamma_T}, \infty)$		
$S_1 \in (-1, S_B)$		$[\sigma_F, \sigma_{H_1}]$	$(\varepsilon_{\gamma_T}, \infty)$	-	$[1, \varepsilon_{\gamma_T})$
		$(\sigma_{H_1}, \sigma_{H_2})$	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T})$
	*	(σ_{H_2}, ∞)	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_T})$	-
	2) $\alpha_{K,K} > 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_F\right)$	$[1, \varepsilon_{\gamma_F})$	-	$(\varepsilon_{\gamma_F}, \varepsilon_{\gamma_T})$
			and $(\varepsilon_{\gamma_T}, \infty)$		
		$[\sigma_F, \sigma_{H_1}]$	$(\varepsilon_{\gamma_T}, \infty)$	-	$[1, \varepsilon_{\gamma_T})$
		(σ_{H_1}, σ_T)	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T})$
		$[\sigma_T, \infty)$	-	$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \infty)$
(iv)	1) $\alpha_{K,K} \leq 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_{H_3}\right)$	$[1, \varepsilon_{\gamma_F})$	-	$(\varepsilon_{\gamma_F}, \varepsilon_{\gamma_T})$
$D_1'(\sigma) < 0$			*and* $(\varepsilon_{\gamma_T}, \infty)$		
$S_1 \in (S_B, 0)$		(σ_{H_3}, σ_F)	$[1, \varepsilon_{\gamma_F})$	$(\varepsilon_{\gamma_F}, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T})$
			and $(\varepsilon_{\gamma_T}, \infty)$		
		$[\sigma_F, \sigma_{H_2})$	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T})$
	*	(σ_{H_2}, ∞)	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_T})$	—
	2) $\alpha_{K,K} > 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_{H_3}\right)$	$[1, \varepsilon_{\gamma_F})$	—	$(\varepsilon_{\gamma_F}, \varepsilon_{\gamma_T})$
			and $(\varepsilon_{\gamma_T}, \infty)$		
		(σ_{H_3}, σ_F)	$[1, \varepsilon_{\gamma_F})$	$(\varepsilon_{\gamma_F}, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T})$
			and $(\varepsilon_{\gamma_T}, \infty)$		
		$[\sigma_F, \sigma_T)$	$(\varepsilon_{\gamma_T}, \infty)$	$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T})$
		$[\sigma_T, \infty)$		$[1, \varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H}, \infty)$

Table 1: Local stability properties and bifurcations

			ε_γ		
Conf		σ	*Saddle*	*Sink*	*Source*
(v)	1) $\alpha_{\Gamma,K}-\alpha_{L,K}\geq 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}},\sigma_{H_3}\right)$	$[1,\varepsilon_{\gamma_F})$	-	$(\varepsilon_{\gamma_F},\varepsilon_{\gamma_T})$
$D_1'(\sigma)>0$			*and* $(\varepsilon_{\gamma_T},\infty)$		
$S_1\in(0,S_D)$		(σ_{H_3},σ_F)	$[1,\varepsilon_{\gamma_F})$	$(\varepsilon_{\gamma_F},\varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H},\varepsilon_{\gamma_T})$
$\alpha_{K,i}=\beta_{K,i}=0$			*and* $(\varepsilon_{\gamma_T},\infty)$		
		(σ_F,σ_{H_2})	$(\varepsilon_{\gamma_T},\infty)$	$[1,\varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H},\varepsilon_{\gamma_T})$
		(σ_{H_2},∞)	$(\varepsilon_{\gamma_T},\infty)$	$[1,\varepsilon_{\gamma_T})$	-
	2) $\alpha_{\Gamma,K}-\alpha_{L,K}< 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}},\sigma_{H_3}\right)$	$[1,\varepsilon_{\gamma_F})$	-	$(\varepsilon_{\gamma_F},\varepsilon_{\gamma_T})$
			and $(\varepsilon_{\gamma_T},\infty)$		
		(σ_{H_3},σ_F)	$[1,\varepsilon_{\gamma_F})$	$(\varepsilon_{\gamma_F},\varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H},\varepsilon_{\gamma_T})$
			and $(\varepsilon_{\gamma_T},\infty)$		
		(σ_F,∞)	$(\varepsilon_{\gamma_T},\infty)$	$[1,\varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H},\varepsilon_{\gamma_T})$
(vi)	1) $\alpha_{\Gamma,K}-\alpha_{L,K}\geq 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}},\sigma^{S_2}\right)$	$[1,\varepsilon_{\gamma_T})$	-	$(\varepsilon_{\gamma_T},\varepsilon_{\gamma_F})$
$D_1'(\sigma)>0$			*and* $(\varepsilon_{\gamma_F},\infty)$		
$S_1\in(S_D,1)$		$\left(\sigma^{S_2},\sigma_F\right)$	$[1,\varepsilon_{\gamma_F})$	$(\varepsilon_{\gamma_F},\varepsilon_{\gamma_T})$	-
$\alpha_{K,i}=\beta_{K,i}=0$			*and* $(\varepsilon_{\gamma_T},\infty)$		
		(σ_F,∞)	$(\varepsilon_{\gamma_T},\infty)$	$[1,\varepsilon_{\gamma_T})$	-
	2) $\alpha_{\Gamma,K}-\alpha_{L,K}< 0$	$\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}},\sigma^{S_2}\right)$	$[1,\varepsilon_{\gamma_T})$	-	$(\varepsilon_{\gamma_T},\varepsilon_{\gamma_F})$
			and $(\varepsilon_{\gamma_T},\infty)$		
		$\left(\sigma^{S_2},\sigma_F\right)$	$[1,\varepsilon_{\gamma_F})$	$(\varepsilon_{\gamma_F},\varepsilon_{\gamma_T})$	-
			and $(\varepsilon_{\gamma_T},\infty)$		
		(σ_F,σ_{H_2})	$(\varepsilon_{\gamma_T},\infty)$	$[1,\varepsilon_{\gamma_T})$	-
		(σ_{H_2},∞)	$(\varepsilon_{\gamma_T},\infty)$	$[1,\varepsilon_{\gamma_H})$	$(\varepsilon_{\gamma_H},\varepsilon_{\gamma_T})$

Table 2: Local stability properties and bifurcations

the individual level ($\varepsilon_\gamma = 1$) may not be appropriate to study the implications of market distortions on local indeterminacy. This is a new interesting result that will be illustrated and further discussed in the economic examples provided in Section 4.

3.1.1 Distortions on the ϱ and Ω functions

We start with the case where distortions affecting only the ϱ and Ω functions are present ($\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$), as it happens, for example, when we only have product or capital market distortions (see Section 4). Then, only configurations $(i) - (iv)$ can be obtained (see (19)), where, as seen above, $\sigma > \sigma_{H_1}$ is a necessary condition for indeterminacy. Note that, although each of these configurations can a priori be obtained, S_1 is always positive when θ is small (see (16)), so that configurations (i) and (ii) are the most relevant ones.

For $\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$, Assumption 2.1, nedeed for indeterminacy when $D_1'(\sigma) < 0$, becomes $\alpha_{L,L} > \theta\alpha_{K,K}$, which has a suitable economic interpretation: the response of effective consumption to labor $(1 + \alpha_{L,L})$ must be stronger than the response of capital income to capital $(1 + \theta\alpha_{K,K})$, when $\sigma = +\infty$. When $\alpha_{K,K} \leq 0$, another necessary condition for indeterminacy is $S_1 \notin (0,1) \Leftrightarrow \frac{1-s-\beta_{K,K}}{s-\beta_{L,L}}\alpha_{L,L} + \alpha_{L,K}\frac{1-s+\beta_{K,L}}{s-\beta_{L,L}} + \alpha_{K,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}} + \alpha_{K,K} > 0$.[22] As a direct implication, under Assumption 1, indeterminacy is ruled out if $\alpha_{L,L}$, $\alpha_{L,K}$, $\alpha_{K,L}$ and $\alpha_{K,K}$ are all negative.

Let us now discuss in detail the role of distortions affecting the ϱ function. For $\alpha_{K,K} > 0$, given the necessary condition $\alpha_{L,L} > \theta\alpha_{K,K}$, indeterminacy requires a positive value for $\alpha_{L,L}$. When $\alpha_{K,K} \leq 0$, and in the absence of distortions on the Ω function ($\alpha_{L,j} = \beta_{L,j} = 0$), the necessary indeterminacy condition $S_1 \notin (0,1)$ requires a positive lower bound on $\alpha_{K,L}$, i.e., $\alpha_{K,L} > -\alpha_{K,K}\frac{1-s+\beta_{K,L}}{1-s-\beta_{K,K}} > 0$, so that a cross effect $(\alpha_{K,L})$ with an opposite sign from the direct effect $(\alpha_{K,K})$ is also necessary. We may then conclude that distortions on the ϱ function, i.e., affecting the capital accumulation equation, do not seem to play a crucial role for indeterminacy. Indeed, indeterminacy either requires opposite effects of capital and labor on the distortions affecting the capital accumulation equation and a distortion due to labor effects $(\alpha_{K,L})$ positive and bounded away from zero, or the presence of some other market failures, distorting the intertemporal trade-off between future consumption and labor.

We now discuss the role of distortions affecting the Ω function. For $\alpha_{L,L} \leq$

[22] Indeed, when $\alpha_{K,K} \leq 0$, indeterminacy is possible in configurations (ii), (iii) and (iv), but does not occur in configuration (i), i.e. it requires $S_1 \notin (0,1)$. This condition is nedeed so that we may have $D > T - 1$, as required for indeterminacy.

0, the necessary condition for indeterminacy ($\alpha_{L,L} > \theta\alpha_{K,K}$) implies $\alpha_{K,K} < 0$, i.e., requires also distortions on the ϱ function. However, this is no longer true for $\alpha_{L,L} > 0$. Indeed, without distortions on the ϱ function, the necessary condition $\alpha_{L,L} > \theta\alpha_{K,K}$ is satisfied for all $\alpha_{L,L} > 0$. Moreover, the other condition $S_1 \notin (0,1)$, necessary when $\alpha_{K,K} = 0$, also holds when the only distortion is an arbitrarily small positive distortion on effective consumption due to labor. Hence indeterminacy may emerge with an arbitrarily small positive $\alpha_{L,L}$.

3.1.2 Distortions on the Ω and Γ functions

We consider now distortions that only affect both Ω and Γ, i.e. the intertemporal trade-off condition of workers, as it happens, for instance, when we have labor market imperfections (see Section 4). In this case $\alpha_{K,i} = \beta_{K,i} = 0$. From (18) and (20), we can see that $D_1'(\sigma)$ can a priori take a positive or negative sign, even for arbitrarily small values of θ.[23] Hence, not only configurations (i) to (iv) are possible, but configurations (v) and (vi) may also emerge. These last two configurations require that $\alpha_{\Gamma L} < \alpha_{\Gamma L}^*$ as shown in (21). They are possible in the absence of distortions on Ω, whereas they never emerge without effects through Γ, since $\alpha_{\Gamma L}^* = -1$ when $\alpha_{\Gamma,K} = \beta_{\Gamma,i} = 0$. Notice, however, that configurations (v) and (vi) require some distortions bounded away from zero.[24] On the contrary, if distortions are arbitrarily weak, $D_1'(\sigma) < 0$ and, when θ is small, S_1 is always positive, so that configurations (i) and (ii) remain the relevant ones.

In the absence of distortions affecting capital accumulation ($\alpha_{K,i} = \beta_{K,i} = 0$), indeterminacy only occurs if $\alpha_{L,L} + \alpha_{L,K} > \alpha_{\Gamma,L} + \alpha_{\Gamma,K}$, a condition with a nice economic interpretation.[25] The term $\alpha_{L,L} + \alpha_{L,K}$ summarizes the global effect of distortions on the Ω function, and $\alpha_{\Gamma,L} + \alpha_{\Gamma,K}$ represents the global effect of distortions in the Γ function, when σ tends to $+\infty$, (see (5)). Therefore, indeterminacy requires a positive difference between these two global distortions on Ω and Γ. This implies that distortions on the generalized offer curve that negatively depend on capital and labor and distortions on the effective consumption that depend positively on capital and labor seem to help the occurrence of indeterminacy. If distortions are sufficiently weak so that $D_1'(\sigma) < 0$, then, by Assumption 2.1, $\alpha_{LL} > \alpha_{\Gamma L}$ is another necessary condition for indeterminacy. Hence, without any effects through labor, i.e.

[23] The same could have been said if we had also considered distortions on the ϱ function.

[24] Note that $\alpha_{\Gamma L}^*$ is close to -1 when $\beta_{\Gamma L}$, $\alpha_{\Gamma K}$ and α_{LL} are close to zero.

[25] Indeed, from Tables 1 and 2, indeterminacy only emerges when either $D_1'(\sigma) < 0$ and $S_1 \notin (0,1)$, or $D_1'(\sigma) > 0$ and $S_1 \in (0,1)$, which implies $\alpha_{L,L} + \alpha_{L,K} > \alpha_{\Gamma,L} + \alpha_{\Gamma,K}$. This condition is needed to ensure $D > T - 1$.

for $\alpha_{\Gamma,L} = \alpha_{L,L} = 0$, indeterminacy is not possible: in the absence of distortions affecting the ϱ function, the effects of distortions through labor are crucial for indeterminacy when distortions are small ($D_1'(\sigma) < 0$).

Using the two necessary conditions mentioned above, and in the absence of distortions on Ω, indeterminacy requires $\alpha_{\Gamma L} < Min\{0, -\alpha_{\Gamma,K}\}$. Thus, indeterminacy is possible when the only distortion is an arbitrarily small negative $\alpha_{\Gamma,L}$.

A few general remarks are now in order. While general distortions on the real interest rate do not seem to play a major role on the occurrence of indeterminacy, distortions on the generalized offer curve and on the effective consumption seem to help the occurrence of indeterminacy. As seen above, indeterminacy is possible in the presence of arbitrarily small distortions affecting either effective consumption or the offer curve. However, this would require infinitely large elasticities of private labor supply and of substitution between inputs.[26] Hence, indeterminacy with plausible values of σ can only occur if distortions take minimal degrees.

4 Applications

We now proceed by applying our general methodology and results to several examples that provide microeconomic foundations for the model developed above. While many of these examples have already been studied in the literature (but not always in a finance constrained Woodford economy) some of them are new, allowing us to exhibit phenomena that have not yet been illustrated. The strategy used to analyze each example is the following. We start by identifying the $\varrho(K, L)$, $\Omega(K, L)$ and $\Gamma(K, L)$ functions.[27] We then compute the elasticities of these three functions with respect to K and L, evaluated at the normalized steady state, and using (5) we identify the parameters $\alpha_{i,j}$ and $\beta_{i,j}$ for $i = K, L, \Gamma$ and $j = K, L$. We can then apply the results obtained in the previous section to discuss the dynamic properties of the economy near the steady state.

We consider four types of examples. We emphasize equivalence results,

[26] Indeed, as seen above, under θ small and weak distortions, only configurations (i) and (ii) are possible (see Table 1). Then, indeterminacy requires $\sigma > \sigma_{H_1}$ and either $\epsilon_\gamma < \epsilon_{\gamma_H}$ or $\epsilon_\gamma < \epsilon_{\gamma_T}$. When distortions become arbitrarily close to zero, σ_{H_1} tends to $+\infty$. In this case, $1/(\epsilon_{\gamma_H} - 1)$ and $1/(\epsilon_{\gamma_T} - 1)$ also tend to $+\infty$. Similar results have been obtained in Ramsey and overlapping generations economies. See Lloyd-Braga et al. (2007) and Pintus (2006), where distortions are due to production externalities.

[27] In all the examples, the functions $A\rho(K/L)$, $A\omega(K/L)$ and $\gamma(L)$, appearing in the definition of $\varrho(K, L)$, $\Omega(K, L)$ and $\Gamma(K, L)$, represent, respectively, the competitive real interest rate, the competitive real wage and the competitive offer curve.

in terms of the indeterminacy mechanisms, across specific distortions fitting within the same type of example. We also discuss the minimal degree of distortions required for indeterminacy to occur with an empirically plausible value of the elasticity of capital-labor substitution (around 1). The first two types of examples have no distortions on the offer curve so that, as discussed above, $D_1'(\sigma) < 0$ (as in Table 1), and, as we shall see, indeterminacy is only obtained under configuration (ii). The last two types of examples illustrate the crucial role played by distortions that affect the offer curve, $\Gamma(K, L)$. We will see in particular that indeterminacy emerges under new configurations (those of Table 1 and Table 2), and that for some sets of distortions' parameters indeterminacy requires a value for ϵ_γ bounded away from 1.

4.1 Examples with the same distortion on the real interest rate and the real wage

In the examples presented below, the same distortion affects both the real interest rate and the real wage, but the generalized offer curve coincides with the competitive one, $\Gamma(K, L) = \gamma(L)$. We will see that in these examples, indeterminacy always requires $\sigma > \sigma_{H_1}$, as an application of Proposition 1(i). We begin with an example where productive externalities are the only distortion. We pursue by explaining that many models with imperfect competition on the product market are, in terms of local dynamic analysis, particular cases of the first example.

Externalities in production have often been introduced in macro-dynamic models (Barinci and Chéron (2001), Benhabib and Farmer (1994), Cazzavillan (2001), Cazzavillan et al. (1998)). In these papers all markets are perfectly competitive and firms face a private constant returns to scale technology, but due **to** positive externalities that affect the total productivity of factors,[28] returns to scale are increasing at the social level. Here, we will extend this formulation, allowing also for negative productive externalities[29] so that, at the social level, returns to scale can be decreasing. We consider therefore that production is given by $y = AF(K, L)\xi(K, L) \equiv ALf(K/L)\xi(K, L)$, where $\xi(K, L)$ stands for externalities, $f(K/L)$ is the usual intensive, strictly increasing and concave, production function. Since firms, when maximizing profits, take externalities as given, we have:

$$\begin{aligned} \Omega_t &= A\omega(K_{t-1}/L_t)\xi(K_{t-1}, L_t) \\ \varrho_t &= A\rho(K_{t-1}/L_t)\xi(K_{t-1}, L_t). \end{aligned}$$

[28] They are usually justified by learning by doing or matching problems on labor market.

[29] These can be justified, for instance, by congestion or pollution arguments.

We obtain $\alpha_{L,L} = \alpha_{K,L} = \varepsilon_{\xi,L}$, $\alpha_{L,K} = \alpha_{K,K} = \varepsilon_{\xi,K}$, and $\alpha_{\Gamma,i} = \beta_{j,i} = 0$, for $i = K, L$ and $j = K, L, \Gamma$, where $\varepsilon_{\xi,i} > -1$ denotes the elasticity of the function $\xi(K, L)$ with respect to $i = K, L$, evaluated at the steady state. It is easy to check that Assumption 4 is satisfied and Assumptions 2 implies $-2 - \nu < (1+\theta)\varepsilon_{\xi,K} < \nu$ where $\nu \equiv \varepsilon_{\xi,L} + \varepsilon_{\xi,K}$. Moreover, Assumption 1 implies that $s(1+\theta\varepsilon_{\xi,K}) > \theta(1+\nu-\varepsilon_{\xi,K})(1-s)$ so that $D_1'(\sigma) < 0$ and only Proposition 1(a), i.e. Table 1, applies.

In the case of negative externalities ($\varepsilon_{\xi,K} < 0$, $\nu - \varepsilon_{\xi,K} < 0$), $S_1 \in (0, 1)$, so that we are in configuration (i). Since $\alpha_{K,K} = \varepsilon_{\xi,K} < 0$, the steady state is always locally determinate (a saddle, see Table 1). Hence negative externalities illustrate the result, discussed in Section 3, that indeterminacy cannot occur when $\alpha_{K,i}$ and $\alpha_{L,i}$ are all negative.

If, as considered for instance in Cazzavillan, Lloyd-Braga and Pintus (1998), we assume positive externalities ($\varepsilon_{\xi,K} > 0$, $\varepsilon_{\xi,L} = \nu - \varepsilon_{\xi,K} > 0$) we obtain $S_1 > 1$. In this case, configuration (ii).1 applies and indeterminacy can emerge for $\sigma > \sigma_{H1} = (s - \theta(1-s))/(\nu - (1+\theta)\varepsilon_{\xi,K})$. Notice that σ_{H_1} can be below one, but this requires sufficiently strong labor externalities. For instance, in the absence of capital externalities ($\varepsilon_{\xi,K} = 0$), indeterminacy emerges in the Cobb-Douglas case if $\nu = \varepsilon_{\xi,L}$ exceed $s - \theta(1-s)$, a value which is too high in empirical terms.

We will now emphasize that, for the analysis of local dynamics, many models with imperfect competition on the product market are, in fact, a particular case of the previous framework with positive productive externalities. Benhabib an Farmer (1994) and Cazzavillan, Lloyd-Braga and Pintus (1998) underlined that this is the case when imperfectly competitive economies are characterized by constant markups and decreasing marginal costs (increasing returns). The same happens when imperfect competition is associated with markup variability or with taste for variety, as we now show. We focus on markup variability linked to strategic interactions between producers and business formation (Dos Santos Ferreira and Lloyd-Braga (2005), Kuhry (2001), Seegmuller (2003, 2008a), Weder (2000a)). We consider that taste for variety is, following Benassy (1996), the consumer utility gain of consuming one unit of all the N_t available varieties of goods instead of consuming N_t units of a single variety (Jacobsen (1998), Seegmuller (2008b)). In these two types of models,[30] increasing returns come from the existence of a fixed cost[31] and the number N_t of producers is determined

[30] For the sake of conciseness, we do not present these models in detail, providing only their main economic features. For more details, the reader can look at the references herein.

[31] Using our notation the production of a firm $i = 1, ..., N_t$ is given by $y_{it} = A(f(k_{it-1}/l_{it})l_{it} - \phi)$, where k_{it-1} (l_{it}) represents capital (labor) used by firm i and $\phi > 0$

by the usual zero profit condition. At equilibrium the number of firms is as an increasing function of aggregate production, i.e. $N_t = N(f(K_{t-1}/L_t)L_t)$, with $\epsilon_N(Y) \equiv N'(Y)Y/N(Y) \geq 0$, and the same distortion $\mu(N_t)$, increasing with N_t, affects both the real wage and the real interest rate, i.e.:

$$\Omega_t = \mu(N_t)A\omega(K_{t-1}/L_t) \tag{9}$$
$$\varrho_t = \mu(N_t)A\rho(K_{t-1}/L_t) \tag{10}$$

where $\epsilon_\mu(N) \equiv \mu'(N)N/\mu(N) \geq 0$. In models with counter-cyclical markups, $\mu(N_t)$ can be interpreted as the inverse of the markup factor, while when there is taste for variety, it represents the ratio between the price set by a single firm and the aggregate price. Substituting the expression for N_t into (9) and (10), we obtain expressions for $\Omega(K, L)$ and $\varrho(K, L)$ similar to the ones of the model with productive externalities. Indeed, defining $\nu \equiv \epsilon_\mu(N)\epsilon_N(Y) \geq 0$, we obtain $\alpha_{L,L} = \alpha_{K,L} = (1-s)\nu$, $\alpha_{L,K} = \alpha_{K,K} = s\nu$ and $\alpha_{\Gamma,i} = \beta_{j,i} = 0$, for $i = K, L$ and $j = K, L, \Gamma$, i.e., for the same value of ν these models correspond to the particular case of the framework with positive productive externalities presented above where $\varepsilon_{\xi,K}/\varepsilon_{\xi,L} = s/(1-s)$. Therefore the previous results apply and indeterminacy can emerge for $\sigma > \sigma_{H1} = (s-\theta(1-s))/[\nu(1-(1+\theta)s)]$.[32] Hence, ν should be high enough to get indeterminacy with a value of σ close to 1, which requires sufficiently counter-cyclical markups (large fixed cost) or a sufficiently high degree of taste for variety.

We can therefore conclude that, in all these examples, indeterminacy may only emerge in the presence of relatively high distortions, like large increasing returns or high markups, that seem to be empirically irrelevant.

4.2 Examples with different distortions on the real interest rate and effective consumption

In this section we present again examples where distortions do not affect the generalized offer curve ($\Gamma(K, L) = \gamma(L)$) so that, as above, $D_1'(\sigma) < 0$ and only Table 1 applies. However, now distortions affecting the real interest rate are different from the ones that modify effective consumption. This will allow us to emphasize that capital market distortions are not, per se, the most relevant ones for indeterminacy. On the contrary, a distortion that only affects effective consumption can be a source of indeterminacy.[33]

is a fixed cost.

[32] Notice that now Assumption 2 is satisfied for $1 - s > \theta s$.

[33] Note however that the degree of this distortion needs to be sufficiently large for indeterminacy to emerge under a not arbitrarily high elasticity of capital-labor substitution.

In the first example presented we consider a perfectly competitive economy where public expenditures, financed by variable taxation under a balanced budget rule, are introduced. We also assume that government spending can affect consumption utility through externalities in preferences. This example follows closely, although extending it by also considering capital income taxation, the work of Lloyd-Braga et al. (2008), and covers as particular cases the fiscal policy rules considered in Dromel and Pintus (2008), Giannitsarou (2005), Guo and Lansing (1998), Gokan (2005), Pintus (2003), Schmitt-Grohé and Uribe (1997), and obviously constant tax rates. The government can levy taxes on capital income ($\rho_t K_{t-1}$), labor income ($\omega_t L_t$) and on private aggregate consumption ($C_t = C_t^w + C_t^c$). Real public spending in goods and services in period t, $G_t \geq 0$, is given by the balanced budget rule $G_t = \tau_L(\omega_t L_t)\omega_t L_t + \tau_C(C_t)C_t + \tau_K(\rho_t K_{t-1})\rho_t K_{t-1}$. Tax rates on labor and capital incomes, and on consumption are respectively given by $\tau_L(\omega_t L_t) \equiv z_L(\omega_t L_t/\omega L)^{\phi_L}$, $\tau_K(\rho_t K_{t-1}) \equiv z_K(\rho_t K_{t-1}/\rho K)^{\phi_K}$ and $\tau_C(C_t) \equiv z_c(C_t/C)^{\phi_c}$, where ωL is the steady state value of the wage bill, ρK the steady state value of capital income and C the steady state level of consumption, while $z_i \in (0,1)$ for $i = L, K$ and $z_c > 0$ represent the level of the tax rates at the steady state. The parameters $\phi_j \in \mathbb{R}$ ($j = L, K, C$) denote the elasticities of the tax rates with respect to the tax bases. When $\phi_j = 0$ the tax rate is constant and equal to z_j. Finally, we denote by $\eta > 0$ the elasticity of public spending externalities in preferences that affect workers' utility of consumption. Assuming that agents take as given the tax rules and externalities, we obtain:

$$\begin{aligned}
\Omega(K_{t-1}, L_t) &= AG_t^\eta \frac{1 - z_L(\omega_t L_t/\omega L)^{\phi_L}}{1 + z_c(C_t/C)^{\phi_c}} \omega(K_{t-1}/L_t) \\
\varrho(K_{t-1}, L_t) &= A[1 - z_K(\rho_t K_{t-1}/\rho K)^{\phi_K}]\rho(K_{t-1}/L_t)
\end{aligned}$$

where G_t is given by the balanced budget condition. In the following, we will analyze separately the effects of each type of taxation on local dynamics.

We start with the case of capital taxation without public spending externalities in preferences, so that market distortions only appear in the function $\varrho(K, L)$. We get $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = L, \Gamma$, $\alpha_{K,K} = -\phi_K \frac{z_K}{1-z_K}$, $\alpha_{K,L} = 0$ and $\beta_{K,K} = -\alpha_{K,K}(1-s) = -\beta_{K,L}$. Obviously, Assumption 4 is satisfied and under θ small, Assumptions 1 and 2 imply that $0 < \phi_K < \frac{(1-z_K)s}{z_K(1-s)}$. We deduce that Proposition 1.(a) applies and that $S_1 \in (0,1)$, so that we are in configuration (i). Since $\alpha_{K,K} < 0$, the steady state is always a saddle (see Table 1). This illustrates that distortions on capital market are not the most powerful to get indeterminacy.

In the case of labor income taxation only, and considering public spending externalities in preferences, distortions only affect effective consumption $\Omega(K,L)$. We have $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = K, \Gamma$, $\alpha_{L,L} = \eta(1+\phi_L) - \phi_L \frac{z_L}{1-z_L}$, $\alpha_{L,K} = 0$ and $\beta_{L,L} = -\alpha_{L,L}s = -\beta_{L,K}$. Assumption 4 is satisfied as before, and Assumptions 1 and 2 imply that $0 < \alpha_{L,L} < 1$, i.e., $-\frac{\eta(1-z_L)}{\eta(1-z_L)-z_L} < \phi_L < \frac{(1-\eta)(1-z_L)}{\eta(1-z_L)-z_L}$ if $\eta > \frac{z_L}{1-z_L}$, or $\frac{(1-\eta)(1-z_L)}{\eta(1-z_L)-z_L} < \phi_L < -\frac{\eta(1-z_L)}{\eta(1-z_L)-z_L}$ if $\eta < \frac{z_L}{1-z_L}$. We deduce that Proposition 1.(a) applies and that configuration (ii).1 is the relevant one[34] so that indeterminacy may emerge if $\sigma > \sigma_{H1} = \frac{(1-z_L)[s(1+\eta)-\theta(1-s)]+\phi_L s[\eta(1-z_L)-z_L]}{\eta(1-z_L)+\phi_L[\eta(1-z_L)-z_L]}$. Therefore, without public spending externalities in preferences ($\eta = 0$), indeterminacy is only possible if $-\frac{1-z_L}{z_L} < \phi_L < 0$, i.e., constant tax rates or tax rates that vary positively with the tax base promote determinacy. Moreover, indeterminacy requires $-\frac{1-z_L}{z_L} < \phi_L < -\frac{1-z_L}{z_L}\frac{s-\theta(1-s)}{(1-s)}$ in the Cobb-Douglas case, i.e., ϕ_L cannot be too close to zero.[35] However, by direct inspection of the expressions of $\alpha_{L,L}$ and σ_{H1}, we notice that these conclusions are no longer valid in the presence of public spending externalities in preferences ($\eta > 0$) where indeterminacy may emerge under a constant or a positively elastic tax rate.

The case of consumption taxation provides a second example of a tax rule that introduces distortions only on effective consumption. Considering for simplicity the case without public spending externalities in preferences ($\eta = 0$), we have $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = K, \Gamma$, $\alpha_{L,L} = -\frac{z_c\phi_c}{1+z_c(1+\phi_c)}\psi$, $\alpha_{L,K} = -\frac{z_c\phi_c}{1+z_c(1+\phi_c)}(1-\psi)$, $\beta_{L,L} = -\alpha_{L,L}\beta s$ and $\beta_{L,K} = -\beta_{L,L}$, where $\psi \equiv \theta(1-s)/[\theta(1-s)+(1-\beta)s]$. Assumption 4 is again satisfied and, for θ sufficiently weak, Assumptions 1 and 2 imply that $1/\beta > \alpha_{L,L} > 0$, i.e., $-\frac{(1+z_c)}{z_c}\frac{1}{1+\beta\psi} < \phi_c < 0$. Using these inequalities, we can deduce that Proposition 1.(a) applies and $S_1 > 1$,[36]. Again configuration (ii).1 is the relevant one, so that indeterminacy may emerge if $\sigma > \sigma_{H1} = \frac{(1+z_c)[s-\theta(1-s)]+z_c\phi_c[s(1-\beta\psi)-\theta(1-s)]}{-z_c\phi_c\psi}$. In the particular case investigated by Giannitsarou (2005) in a Ramsey model where government spending is constant ($\phi_c = -1$), indeterminacy occurs under a Cobb-Douglas technology provided z_c is larger than a lower bound ($z_c > [s-\theta(1-s)]/(1-s\beta)\psi$).

These two last examples, based on fiscal policy rules and public spending externalities in preferences, show that, although indeterminacy may emerge

[34] We assume that $2[s-\theta(1-s)] > \theta(1-s)$ which implicitly requires a sufficiently weak θ.

[35] Similarly, if we fix the value of $\phi_L < 0$, indeterminacy requires a sufficiently high value for z_L. For example, when $\eta = 0$ and $\phi_L = -1$ (the case considered in Schmitt-Grohé and Uribe (1997), Pintus (2003) and Gokan (2005) of a constant government spending), indeterminacy only emerges for all $\sigma \geq 1$ if $z_L > [s-\theta(1-s)]/[1-\theta(1-s)]$.

[36] As before, this requires a sufficiently weak θ.

when distortions only modify effective consumption, it only emerges for plausible values of the elasticity of capital-labor substitution in the presence of sufficiently large distortions (taxes in this case).

We end this section by showing that models with consumption externalities in preferences (Alonso-Carrera et al. (2008), Gali (1994), Ljungqvist and Uhlig (2000), Weder (2000b)) introduce market distortions that again only affect $\Omega(K, L)$, and are in fact perfectly equivalent to the previous examples with labor income or consumption taxation.

Consumption externalities correspond to the idea that individual utility of consumption is affected by the current consumption of others (envy or altruism), so that aggregate or average consumption becomes an argument of the utility function. In our framework, this amounts to consider that workers' utility is given by $U(C_{t+1}^w \varphi(\overline{C}_{t+1})/B) - V(L_t)$, where $\overline{C}_{t+1}$ denotes average consumption and $\varphi(\overline{C}_{t+1})$ the externality function.[37] Accordingly, at equilibrium,

$$\Omega(K_{t-1}, L_t) = A\varphi(C_t)\omega(K_{t-1}/L_t)$$

Since $\varrho(K_{t-1}, L_t)$ and $\Gamma(K_{t-1,}L_t)$ are not affected by this distortion, $\alpha_{j,i} = \beta_{j,i} = 0$ for $i = K, L$ and $j = K, \Gamma$. Let υ be the elasticity of φ with respect to C, evaluated at the steady state. To fully determine $\Omega(K_{t-1}, L_t)$, we have to precise whether individual workers compare themselves to the average worker or to the average consumer, i.e., whether $C = C^w$ or $C = C^w + C^c$.

When $C = C^w$, we have $\alpha_{L,L} = \upsilon$, $\alpha_{L,K} = 0$ and $\beta_{L,L} = -\alpha_{L,L}s = -\beta_{L,K}$. Note that the same values for $\alpha_{j,i}$ and $\beta_{j,i}$ can be obtained by considering instead the case of labor income taxation presented above, if we take $\upsilon = \eta(1+\phi_L) - \phi_L \frac{z_L}{1-z_L}$. Hence, from the point of view of local indeterminacy and dynamics, the two models are perfectly similar, i.e, the mechanisms operating for indeterminacy are the same, even if their economic interpretations are different.[38]

When $C = C^w + C^c$, the results are slightly different. Now $\alpha_{L,L} = \upsilon\psi$, $\alpha_{L,K} = \upsilon(1-\psi)$ and $\beta_{L,L} = -\alpha_{L,L}s\beta = -\beta_{L,K}$, with $\psi = \theta(1-s)/[(1-\beta)s+\theta(1-s)]$. There is now an equivalence with the model with consumption taxation since for $\upsilon = -\frac{z_c\phi_c}{1+z_c(1+\phi_c)}$ the parameters $\alpha_{j,i}$ and $\beta_{j,i}$ are identical in both models.[39] Hence, once more, although the distortions introduced in these models have different microeconomic foundations, they operate exactly in the same way.

[37] We do not introduce externalities into capitalists preferences because, since they have a log-linear utility function, such externalities would not affect the dynamics.

[38] Notice that this implies that indeterminacy only occurs when $\upsilon > 0$, i.e., consumption externalities are of the "keeping-up with the Joneses" type.

[39] Again, indeterminacy requires consumption externalities of the "keeping-up with the Joneses" type.

With these examples we were able to illustrate the fact that distortions that affect the capital market do not promote indeterminacy. Indeed, when there is only capital income taxation, the steady state is always a saddle. On the contrary, when distortions only modify effective consumption (labor income taxation, public spending externalities, consumption taxation), indeterminacy can emerge. As in the first type of examples, configuration (ii) of Table 1 applies. Therefore, the mechanisms for indeterminacy are quite similar, even if the economic interpretation of distortions across all these examples is different. Finally, we showed that models where aggregate consumption affects consumption utility (consumption externalities) are perfectly equivalent, from a local dynamic point of view, to models with labor income and consumption taxation.

4.3 Examples with distortions only on the generalized offer curve

In the examples presented in this section distortions only affect the generalized offer curve. We begin with the case of leisure externalities in preferences and show that, with respect to the previous examples, several new dynamic configurations and results emerge, depending on the level of this externality. We pursue by presenting an example with efficiency wages and establish that, in terms of local dynamics, it corresponds to a particular case of the model with leisure externalities.

The idea behind leisure externalities is that an individual's utility from leisure is affected by the amount of labour supplied by others. For the sake of simplicity, let the utility function of a worker be written as $C^w_{t+1}/B - \overline{L}_t^{\mu} L_t^{\epsilon_\gamma}$, where $\epsilon_\gamma \geq 1$, $\mu \in \mathbb{R}$ are constant parameters and $\overline{L}_t$ denotes aggregate labor, which is taken as given by individual workers, but modifies their welfare. Solving the model, and since at equilibrium $L_t = \overline{L}_t$, we get $\alpha_{j,i} = \beta_{j,i} = 0$ and $\beta_{\Gamma,i} = \alpha_{\Gamma,K} = 0$ for $\{i,j\} = \{K,L\}$, and $\alpha_{\Gamma,L} = \mu$, i.e. distortions only affect $\Gamma(K,L)$ through the parameter $\alpha_{\Gamma,L}$. We can easily see that Assumptions 1 and 4 are ensured. Moreover, from Assumption 2, we focus on cases where $-2 < \mu < 0$. Note that when $\mu < 0$ the marginal desutility of labor is lower when the others also work more.[40]

Let us define $\mu^* \equiv [2[s-\theta(1-s)] - 2\sqrt{s[s-\theta(1-s)]}]/[\theta(1-s)]$. If $\mu^* < \mu < 0$, configurations (ii).1 and (iii).1 of Table 1 apply, meaning that indeterminacy requires $\sigma > \sigma_{H1}$. Since $\sigma_{H1} < 1$ when $\mu < -[s-\theta(1-s)]/[1-\theta(1-s)]$, indeterminacy can occur when the production function is

[40]Recent contributions that have introduced leisure externalities (Benhabib and Farmer (2000), Weder (2004)) also assumed $\mu < 0$.

Cobb-Douglas for $\mu^* < \mu < -[s - \theta(1-s)]/[1-\theta(1-s)]$.

For $\mu < \mu^*$, indeterminacy emerges for a different range of the elasticity of capital-labor substitution. Indeed, if $-1 < \mu < \mu^*$, configuration (iv).1 applies, the steady state being locally indeterminate for $\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$ when $\sigma_{H3} < \sigma < \sigma_F$, for $1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_H}$ when $\sigma_F < \sigma < \sigma_{H_2}$, and for $1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$ when $\sigma > \sigma_{H_2}$. See Table 1.

If $-2 < \mu < -1$,[41] configuration (v).1 applies. The conditions for indeterminacy are qualitatively similar to those obtained just before but quantitatively, the results are different. Indeed, as $\sigma_F = \frac{2s+(1-s)\theta(2+\mu)}{2(2+\mu)}$, we deduce that $\sigma_F < 1$ for $\mu > \mu^{**} \equiv \frac{-2[2-s-\theta(1-s)]}{2-\theta(1-s)}$, with $-2 < \mu^{**} < -1$. While for $-1 < \mu < \mu^*$, indeterminacy emerges under a Cobb-Douglas technology when the private labor supply is infinitely elastic, we do not get this result for all $-2 < \mu < -1$. More precisely, when $-2 < \mu < \mu^{**}$,[42] indeterminacy may be ruled out because private labor supply is too elastic ($\varepsilon_{\gamma_F} > \varepsilon_\gamma \geq 1$). This shows that it should be misleading to focus only on the case of an infinitely elastic labor supply when we want to fully study the role of some labor market distortions on the occurrence of indeterminacy.

We end this section by presenting an example with efficiency wages and show that, in terms of local dynamics, such a framework can be seen as a particular case of the previous model with leisure externalities. The example we present here follows closely Grandmont (2008), where unemployment insurance[43] is introduced in a Woodford economy with efficiency wages (see also (Coimbra (1999), Nakajima (2006)). At equilibrium, these distortions only affect the generalized offer curve: $\Gamma(K_{t-1}, L_t) = g(L_t)$, where $g(L)$ stands for aggregate consumption of employed and unemployed workers, and its elasticity satisfy $0 < \epsilon_g < 1$, with ϵ_g close to 1 when unemployment benefits are small, and decreasing to zero when these benefits become larger. Because of a constant reservation wage,[44] we have $\epsilon_\gamma = 1$. Hence $-1 < \alpha_{\Gamma,L} = (\epsilon_g - 1) < 0$, and $\alpha_{i,j} = \beta_{i,j} = 0$ for $\{i,j\} = \{K,L\}$, $\beta_{\Gamma,L} = \beta_{\Gamma,K} = \alpha_{\Gamma,K} = 0$.

Comparing the parameters $\alpha_{i,j}$ and $\beta_{i,j}$ of this economy with those of the leisure externalities example presented above, we can see that they are formally identical for $\mu = \epsilon_g - 1$, provided $-1 < \mu < 0$ and $\epsilon_\gamma = 1$. Therefore, this last model can be seen as a particular case of not too negative aggregate labor externalities in leisure utility. Recall that for $\mu^* < \mu < 0$, configurations

[41] Notice that in their paper, Benhabib and Farmer (2000) set $\mu = -1.23$.

[42] This also requires $\sigma_{H3} < 1$.

[43] Unemployment compensation is a constant percentage of wages, financed by taxation on labor income at a uniform rate.

[44] Each worker supplies one unit of labor with a labor desutility that depends on the level of effort. Since at equilibrium the level of effort is constant, individual labor supply is infinitely elastic.

(ii).1 and (iii).1 of Table 1 apply, i.e., indeterminacy emerges for $\sigma > \sigma_{H1}$. Because $\sigma_{H1} < 1$ requires $0 < \epsilon_g < (1-s)/[1-\theta(1-s)]$,[45] unemployment insurance cannot be arbitrarily weak when the technology is Cobb-Douglas. However, the range of values for ϵ_g such that indeterminacy emerges for all $\sigma \geq 1$ is compatible with a wide and quite plausible range of values of unemployment and unemployment insurance rates.[46]

4.4 Examples with distortions on the generalized offer curve and effective consumption

In this section we focus on examples where both $\Gamma(K, L)$ and $\Omega(K, L)$ are affected by market distortions, while the capital market remains perfectly competitive. The first example deals with labor market imperfections: unions and unemployment benefits are introduced. In the second example, public spending, financed through a balanced budget rule, affects both individual utility of consumption and labor desutility. This case is new, and is able to illustrate most of the dynamic results (configurations) exhibited in the general framework.

The first example is based on Dufourt et al. (2008), where the Woodford finance constrained framework is extended to take into account the existence of involuntary unemployment (see also Lloyd-Braga and Modesto (2007)). In this paper, wages and employment are determined through an efficient bargain between unions and firms. Unions are able to set wages above a reservation wage, with a markup factor $\mu(K, L) = \frac{1-\alpha s(K/L)}{1-s(K/L)} \geq 1$, increasing in the (constant) bargaining power of unions $(1-\alpha) \in [0, 1)$.[47] Employment is determined by the equality between the reservation wage and the marginal productivity of labor.Dufourt et al. (2008) also consider a constant real unemployment benefit $b > 0$ financed by taxes on those employed. Identifying their model with our general framework, we obtain:

$$\begin{aligned}
\Omega(K_{t-1}, L_t) &= A\mu(K_{t-1}/L_t)\omega(K_{t-1}/L_t) \\
\varrho(K_{t-1}, L_t) &= A\alpha\rho(K_{t-1}/L_t) \\
\Gamma(K_{t-1,}L_t) &= b\frac{\mu(K_{t-1}/L_t)}{L_t}
\end{aligned}$$

[45] From the previous example on leisure externalities, remember that $\sigma_{H1} < 1$ requires $\mu < -[s-\theta(1-s)]/[1-\theta(1-s)]$.

[46] See Grandmont (2008) for a more detailed discussion.

[47] Note that the case of a perfectly competitive labor market would be obtained with $\mu(K, L) = 1$, i.e., $\alpha = 1$.

Because of the existence of a reservation wage[48] we have $\epsilon_\gamma = 1$ and, after some computations, we get $\alpha_{K,j} = \beta_{K,j} = 0$ for $j = K, L$,[49] $\alpha_{\Gamma,K} = \alpha_{L,K} = -\alpha_{L,L} = -\beta_{\Gamma,K} = \beta_{L,L} = -\beta_{L,K} = \beta_{\Gamma,L} = \frac{s(1-\alpha)}{1-\alpha s} \in (0,s)$, and $\alpha_{\Gamma,L} = \alpha_{L,L} - 1$. Note that Assumption 2 is satisfied for $s < 1/2$, and Assumption 1 for $\alpha s > \theta(1-s)$.

One can check that, for $\theta(1-s)/s < \alpha < 1 - [\theta(1-s)/s(4-\theta)]$, configurations $(ii).1$ and $(iii).1$ are the relevant ones and indeterminacy emerges for $\sigma > \sigma_{H1} = s$ (see Table 1). For $1 - [\theta(1-s)/s(4-\theta)] < \alpha \leq 1$, configuration $(iv).1$ applies. Since $\epsilon_\gamma = 1$, indeterminacy emerges when $\sigma > \sigma_F = \frac{2s[\alpha(2-s)-1]+\theta(1-s)(1-\alpha s)}{2[1-s(2-\alpha)]}$. Notice that for $1 - [\theta(1-s)/s(4-\theta)] < \alpha \leq 1$, we have $s < \sigma_F < 1$. Therefore, the steady state is always indeterminate if the technology is Cobb-Douglas, even when union power is arbitrarily small or equal to zero.[50] Therefore, and recalling also the efficiency wages example, we conclude that with plausible labor market imperfections, indeterminacy emerges under reasonable degrees of capital-labor substitution.[51]

In the last example, the standard Woodford model is modified introducing public spending, financed by taxation on capital and labor incomes through a balanced budget rule. Assuming that government expenditures (G_t) provide services that affect not only workers' utility for consumption, but also their desutility of labor, we will be able to provide an illustration of configuration (vi). For simplicity, let the utility of the representative worker be defined by $G_{t+1}^{\eta} C_{t+1}^{w}/B - G_t^{\mu} L_t^{\epsilon_\gamma}$, where $\epsilon_\gamma \geq 1$ and η and μ are parameters, representing, respectively, the elasticity of government spending affecting consumption utility and labor desutility. The level of government spending is defined by $G_t = \tau_K \rho_t K_{t-1} + \tau_L \omega_t L_t$, where $\tau_K \in (0,1)$ and $\tau_L \in (0,1)$ denote, respectively, the capital and labor income tax rates that are supposed to be constant. At equilibrium, we obtain:

$$\begin{aligned}
\Omega(K_{t-1}, L_t) &= [\tau_K \rho_t K_{t-1} + \tau_L \omega_t L_t]^{\eta} (1-\tau_L)\omega(K_{t-1}/L_t) \\
\varrho(K_{t-1}, L_t) &= (1-\tau_K) A \rho(K_{t-1}/L_t) \\
\Gamma(K_{t-1}, L_t) &= [\tau_K \rho_t K_{t-1} + \tau_L \omega_t L_t]^{\mu} \gamma(L_t),
\end{aligned}$$

where $\gamma(L_t) = \epsilon_\gamma L_t^{\epsilon_\gamma}$, $\epsilon_\gamma - 1 \geq 0$ representing the inverse of the elasticity of

[48] Each worker supplies inelastically 1 unit of labor. Due to the unemployment benefit, there is a reservation wage below which individuals prefer not to work.

[49] This means that, in terms of local dynamics, the model is as if $\varrho(K_{t-1}, L_t)$ is not affected by distortions.

[50] This is due to the significant difference between the parameters $\alpha_{L,L}$ and $\alpha_{\Gamma,L}$ coming from the existence of the unemployment benefit.

[51] Indeed, in both cases, indeterminacy is implied by the existence of sufficiently negative values of $\alpha_{\Gamma,L}$ (ϵ_g low in the efficiency wage model, $\alpha_{\Gamma,L} < -1$ here), that correspond to plausible values of the replacement ratio and of the unemployment rate.

the labor supply at the individual level. By direct inspection of $\varrho(K_{t-1}, L_t)$, we immediately deduce that $\alpha_{K,i} = \beta_{K,i} = 0$.[52] Let us define $\psi \equiv \tau_L(1-s)/[\tau_L(1-s)+s\tau_K] \in (0,1)$. We get $\alpha_{L,L} = \eta\psi$, $\alpha_{L,K} = \eta(1-\psi)$, $\beta_{L,L} = (1-s-\psi)\eta = -\beta_{L,K}$, and $\alpha_{\Gamma,L} = \mu\psi$, $\alpha_{\Gamma,K} = \mu(1-\psi)$, $\beta_{\Gamma,L} = (1-s-\psi)\mu = -\beta_{\Gamma,K}$.

In order to simplify the presentation, we consider that $\eta > \max\{\mu, 0\}$[53] and $\psi < 1 - s(1+\eta)/(1-s+\eta) < 1-s$. In this case, Assumptions 1 and 2 further require that $-2/\psi - \mu < \eta < s/(1-s-\psi)$, $\mu > -s/(1-s-\psi)$, and that $\theta < \theta^* \equiv [s-\eta(1-s-\psi)]/(1+\eta\psi)(1-s)$.

Let $\mu^b \equiv \frac{(1+\eta\psi)(\theta-\theta^*)}{1-\psi-\theta(1+\eta\psi)}$. For $\mu > \mu^b$, $D_1'(\sigma) < 0$ and configurations (ii).1, (iii).1 and (iv).1 are the relevant ones. In configurations (ii).1 and (iii).1 indeterminacy emerges for $\sigma > \sigma_{H1} = \frac{s-(1-s-\psi)(\eta-\mu)-\theta(1-s)(1+\mu)}{\psi(\eta-\mu)}$, provided ε_γ is sufficiently small. See Table 1.

In configuration (iv).1, which emerges when μ is sufficiently close to μ^b, indeterminacy occurs for $\sigma_{H_3} < \sigma < \sigma_F$ when $\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$, and for $\sigma > \sigma_F$ when $1 \leq \varepsilon_\gamma < \min\{\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T}\}$, where $\sigma_F = \frac{2[s-(1-s-\psi)(\eta+\mu)]+\theta(1-s)(2+\eta+\mu)}{2[2+\psi(\eta+\mu)]}$. Hence, when the capital-labor substitution σ is not too large, indeterminacy may be ruled out if ε_γ is sufficiently close to one. When $\mu < \mu^b$, $D_1'(\sigma) > 0$. So either configuration (v).2 applies[54] and indeterminacy occurs under similar conditions than under configuration (iv).1, or configuration (vi).2 is the relevant one and indeterminacy occurs for $\sigma^{S_2} < \sigma < \sigma_F$ when $\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma_T}$, and for $\sigma > \sigma^F$ when $1 \leq \varepsilon_\gamma < \min\{\varepsilon_{\gamma_H}, \varepsilon_{\gamma_T}\}$.[55] This example shows that distortions that affect both sides of the intertemporal trade-off of workers, i.e., effective consumption $\Omega(K, L)$ and the generalized offer curve $\Gamma(K, L)$, are able to generate most of the possible dynamic configurations exhibited in our general framework.

5 Concluding remarks

With our general analysis of the role of market distortions on local dynamics and the different examples of specific distortions presented above, we were able to we emphasize several interesting results, some of them already latent in previous works, but which are here confirmed, generalized and highlighted.

[52] In terms of local dynamics this means that government intervention does not affect $\varrho(K_{t-1}, L_t)$, i.e., this model is equivalent to a model where $\varrho(K_{t-1}, L_t) = \rho_t$.

[53] Note that since $\alpha_{K,i} = \beta_{K,i} = 0$, $\eta > \mu$ is a necessary condition for indeterminacy.

[54] Note that $\alpha_{\Gamma,K} - \alpha_{L,K} = -(\eta-\mu)(1-\psi) < 0$.

[55] Configuration (v) emerges for $\mu^c < \mu < \mu^b$, while for $\mu < \mu^c$ we obtain configuration (vi), where $\mu^c \equiv \frac{(1-s)(1+\eta)(1+\eta\psi)(\theta-\theta^*)}{(1-s)(1-\psi)-s+(1-s-\psi)\eta} < 0$. Note that μ^c falls into the appropriate range of values for μ if η is sufficiently large and ψ is close to $1-s$.

First of all, capital market distortions per se do not seem to play a major role for the occurrence of indeterminacy. On the contrary, bifurcations and indeterminacy emerge under labor market rigidities, without imposing strange or implausible restrictions, whereas for output market distortions, indeterminacy requires conditions that might be considered less relevant from an empirical point of view. These findings suggest that the functioning of labor markets, which in the real world show significant deviations from the competitive paradigm, may be responsible for the persistency along business fluctuations and for the existence of expectations-driven cycles. Empirical analysis on this issue is therefore an important direction for further research.[56]

A possible explanation for these results may be linked to the fact that future expectations, which open the room for fluctuations driven by self-fulfilling expectations, only affect the current decisions of consumers/workers, thus rendering distortions that affect the intertemporal trade-off of consumers/workers more important than those affecting the capital accumulation equation.[57] Strategic considerations by firms owning productive capital, which are usually disregarded, may render future expectations of capitalists/producers relevant, and change the results. Although some works have already considered some of these aspects,[58] further research on this issue is welcome.

Two remarks concerning the scope of the present study are worth referring. First, our paper only deals with the role of market distortions on local indeterminacy linked to the sink property, i.e. we do not adress the cases of static or global dynamic indeterminacy and bifurcations,[59] which may also appear in the presence of some market imperfections. Second, although we only discuss local deterministic indeterminacy and cycles, we may be able to construct stochastic sunspot cycles along indeterminacy and bifurcations

[56] Some recent works confirm the importance of labor market imperfections in explaining real business cycles data. Dufourt et al. (2007) replicate the fluctuations and persistence of unemployment data, considering i.i.d. sunspot shocks on expecations in a model with unions and unemployment benefits. Chari et al. (2007) also show quantitatively that labor market frictions constitute one of the most promising mechanisms through which shocks on fundamentals lead to business cycles fluctuations.

[57] Indeed, in all the usual macrodynamic frameworks, including the Ramsey and overlapping generations models, firms just rent productive capital, accummulated from past savings of consumers/capitalists, so that future expectations do not directly influence capital accumulation.

[58] See for instance d'Aspremont et. al (2000).

[59] The interested reader can refer to Dos Santos Ferreira and Dufourt (2006), Dos Santos Ferreira and Lloyd-Braga (2008) and Wang and Wen (2008) who exploit static indeterminacy of the equilibrium to explain expectation-driven fluctuations and to Drugeon and Wigniolle (1996) and Gali (1995) who obtain multiplicities from the analysis of global dynamics.

using the methodology of Grandmont et al. (1998). However, under some market failures, with strategic interactions between agents, our parameters α_{ij} and β_{ij} may become stochastic in the presence of extrinsic uncertainty and, in this case, the results provided in Grandmont et al. (1998) may no longer be applicable, in the context of autocorrelated sunspot processes, along bifurcations.

Finally, let us notice that our work enabled us to find classes of specific distortions within which equivalence results are obtained. This has some strong implications. If we estimate the relevant parameters of our general formulation, we will not be able to identify a particular source of specific distortions among those, which belonging to the same class, are observationally equivalent. Also, even if indeterminacy requires an empirically unreasonable degree of some specific distortion, the associated indeterminacy mechanism is not necessarily unimportant, since an equivalent empirically plausible model may exist.

6 Appendix

6.1 Existence of a steady state

Proposition 2 (Existence of the normalized steady state) $(K^*, L^*) = (1,1)$ *is a stationary solution of the dynamic system (3)-(4) if and only if* $A = \theta/(\beta\varrho(1,1) > 0$ *and* $B = [\beta\varrho(1,1)\Gamma(1,1)]^{-1}\theta\Omega(1,1) > 0$.

Proof. A stationary equilibrium of the dynamic system (3)-(4) is a solution $(K, L) = (K_{t-1}, L_t)$ for all t, that satisfies $A\varrho(K, L) = \theta/\beta$ and $A/B)\Omega(K, L)L = \Gamma(K, L)$. The existence of a steady state can be established by choosing appropriately the two scaling parameters $A > 0$ and $B > 0$ so as to ensure that one steady state coincides with $(K, L) = (1, 1)$. From the first equation, we obtain a unique solution $A = \theta/(\beta\varrho(1,1) > 0$. Substituting this into the second equation we then obtain the unique solution for $B = [\beta\varrho(1,1)\Gamma(1,1)]^{-1}\theta\Omega(1,1) > 0$.

6.2 Trace T and determinant D of the Jacobian matrix

The trace T and the determinant D of the associated Jacobian matrix J, defined in (6), are given respectively, by $T = 1 + \frac{\varepsilon_{\Gamma,L}+\theta(\varepsilon_{\varrho,K}(1+\varepsilon_{\Omega,L})-\varepsilon_{\Omega,K}\varepsilon_{\varrho,L})}{1+\varepsilon_{\Omega,L}}$ and $D = \frac{\varepsilon_{\Gamma,L}(1+\theta\varepsilon_{\varrho,K})-\theta\varepsilon_{\Gamma,K}\varepsilon_{\varrho,L}}{1+\varepsilon_{\Omega,L}}$. We substitute the expressions given in (5) in these two equations, and we assume that the numerator and the denominator of T and D linearly depend on the elasticity of capital-labor substitution σ, i.e. that:

Assumption 3
$(\beta_{L,K} + s) = \frac{(1-s-\beta_{K,K})(s-\beta_{L,L})}{(1-s+\beta_{K,L})}$ *and* $\beta_{\Gamma,K} = -\beta_{\Gamma,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}$.

This assumption is satisfied in models with no distortion and by all the works considered in the literature and presented here as applications. Under Assumption 3, T and D can then be written as:

$$\begin{aligned}
T &= T_0(\sigma)(\varepsilon_\gamma - 1) + T_1(\sigma), \quad T_0(\sigma) = \frac{\sigma}{\sigma(1+\alpha_{L,L}) - (s - \beta_{L,L})} \\
T_1(\sigma) &= 1 + \{\sigma[1 + \alpha_{\Gamma,L} + \theta(\alpha_{K,K}(1+\alpha_{L,L}) - \alpha_{L,K}\alpha_{K,L})] + \beta_{\Gamma,L} \\
&\quad - \theta[(1+\alpha_{L,L})(1-s-\beta_{K,K}) + \alpha_{K,K}(s-\beta_{L,L}) + \alpha_{L,K}(1-s+\beta_{K,L}) \\
&\quad + \alpha_{K,L}\frac{(1-s-\beta_{K,K})(s-\beta_{L,L})}{1-s+\beta_{K,L}}]\}/\{\sigma(1+\alpha_{L,L}) - (s-\beta_{L,L})\}
\end{aligned} \tag{11}$$

$$\begin{aligned}
D &= D_0(\sigma)(\varepsilon_\gamma - 1) + D_1(\sigma), \quad D_0(\sigma) = \frac{\sigma(1+\theta\alpha_{K,K}) - \theta(1-s-\beta_{K,K})}{\sigma(1+\alpha_{L,L}) - (s-\beta_{L,L})} \\
D_1(\sigma) &= \{\sigma[(1+\theta\alpha_{K,K})(1+\alpha_{\Gamma,L}) - \theta\alpha_{\Gamma,K}\alpha_{K,L}] + \beta_{\Gamma,L}(1+\theta\alpha_{K,K}) \\
&\quad - \theta[(1-s-\beta_{K,K})(1+\alpha_{\Gamma,L}) + \alpha_{\Gamma,K}(1-s+\beta_{K,L}) \\
&\quad - \alpha_{K,L}\beta_{\Gamma,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}]\}/\{\sigma(1+\alpha_{L,L}) - (s-\beta_{L,L})\}
\end{aligned} \tag{12}$$

6.3 Local stability properties

Our main interest is to understand how the existence of market imperfections change the characterization of local stability properties in terms of the elasticity of substitution between capital and labor, σ, and of the elasticity of the private offer curve, ε_γ, while keeping s and θ constant at values satisfying Assumption 1.1. Applying the geometrical method developed by Grandmont et al. (1998), we start by analyzing the variations of T and D in the plane (T, D), as ε_γ is running the interval $[1, +\infty)$. From (11) and (12), we see that, in the plane (T, D), the locus of points $(T(\varepsilon_\gamma), D(\varepsilon_\gamma))$ describes a half-line Δ for $\varepsilon_\gamma \in [1, +\infty)$, that starts at $(T_1(\sigma), D_1(\sigma))$ when $\varepsilon_\gamma = 1$, with a slope S equal to

$$S = D_0(\sigma)/T_0(\sigma) = 1 + \theta\alpha_{K,K} - \theta\left(1 - s - \beta_{K,K}\right)\sigma^{-1} > 0 \qquad (13)$$

Under Assumption 1, $D_0(\sigma) > 0$ and $T_0(\sigma) > 0$, i.e. the half-line Δ is positively sloped, pointing upwards to the right, as ε_γ increases from 1 to $+\infty$. When σ goes from $(s-\beta_{LL})/(1+\alpha_{LL})$ to $+\infty$, S increases to $1+\theta\alpha_{K,K}$. Easy analytical computations show that:

Lemma 1 *Under Assumption 1 if $\alpha_{KK} \leq 0$ then $0 < S < 1$, whereas if $\alpha_{KK} > 0$, then $1 + \theta\alpha_{K,K} > S > 1$ if and only if $\sigma > \sigma_T$, and $S = 1$ if and only if $\sigma = \sigma_T$, with $\sigma_T \equiv (1 - s - \beta_{KK})/\alpha_{KK}$.*

To locate the half-line Δ in the plane (T, D), it is also important to know, not only its slope but also the position of its starting point $(T_1(\sigma), D_1(\sigma))$ as σ varies from $+\infty$ to $(s-\beta_{LL})/(1+\alpha_{LL})$. It describes a half-line Δ_1, starting at $(T_1(+\infty), D_1(+\infty))$ given by:

$$\begin{aligned} T_1(+\infty) &= 1+\left[1+\alpha_{\Gamma,L}+\theta(\alpha_{K,K}(1+\alpha_{L,L})-\alpha_{L,K}\alpha_{K,L})\right]\left[1+\alpha_{L,L}\right]^{-1} && (14) \\ D_1(+\infty) &= \left[1+\alpha_{\Gamma,L}+\theta\left(\alpha_{K,K}(1+\alpha_{\Gamma,L})-\alpha_{\Gamma,K}\alpha_{K,L}\right)\right]\left[1+\alpha_{L,L}\right]^{-1} && (15) \end{aligned}$$

We focus on configurations where this starting point lies on the line (AC), i.e. we assume that:

Assumption 4
$1 + D_1(+\infty) - T_1(+\infty) = 0$ *i.e.*, $\alpha_{K,K}(\alpha_{\Gamma,L} - \alpha_{L,L}) = \alpha_{K,L}(\alpha_{\Gamma,K} - \alpha_{L,K})$.

Most of the distortions considered in the literature satisfy this condition. Note that from Assumption 2 $(T_1(+\infty), D_1(+\infty))$ lies between A an C.

Under Assumption 4 the slope S_1 of the half-line Δ_1, is given by:

$$S_1 = \frac{D_1'(\sigma)}{T_1'(\sigma)} = 1 + \theta\frac{I_2}{I_1}, \qquad (16)$$

with

$$\begin{aligned} I_1 \equiv\ & \theta(1+\alpha_{L,L})[(1+\alpha_{L,L})(1-s-\beta_{K,K}) + \alpha_{L,K}(1-s+\beta_{K,L}) \\ & +\alpha_{K,L}\frac{(1-s-\beta_{K,K})(s-\beta_{L,L})}{1-s+\beta_{K,L}}] \\ & -(s-\beta_{L,L})[1+\alpha_{\Gamma,L}-\theta\alpha_{L,K}\alpha_{K,L}] - (1+\alpha_{L,L})\beta_{\Gamma,L}; \qquad (17) \\ I_2 \equiv\ & -(1+\alpha_{L,L})[(1-s-\beta_{K,K})\left(\alpha_{L,L}-\alpha_{\Gamma,L}\right) \\ & +\alpha_{K,K}(s-\beta_{L,L}+\beta_{\Gamma,L}) + (\alpha_{L,K}-\alpha_{\Gamma,K})(1-s+\beta_{K,L}) \\ & +\alpha_{K,L}\frac{(1-s-\beta_{K,K})(s-\beta_{L,L}+\beta_{\Gamma,L})}{1-s+\beta_{K,L}}]. \end{aligned}$$

As σ decreases from $+\infty$ to $(s-\beta_{LL})/(1+\alpha_{LL})$, the half-line Δ_1 may point upwards ($D_1'(\sigma) < 0$) or downwards ($D_1'(\sigma) > 0$).

6.4 Configurations

We analyze separately these two cases, $D_1'(\sigma) < 0$ and $D_1'(\sigma) > 0$. Note that:

$$SignD_1'(\sigma) = Sign(I_1 + \theta I_2), \tag{18}$$

where

$$\begin{aligned} I_1 + \theta I_2 \equiv\ & \theta(1+\alpha_{L,L})[(1-s-\beta_{K,K}) + (1-s-\beta_{K,K})\alpha_{\Gamma,L} \\ & -\alpha_{K,K}(s-\beta_{L,L}+\beta_{\Gamma,L}) + \alpha_{\Gamma,K}(1-s+\beta_{K,L}) \\ & -\alpha_{K,L}\frac{(1-s-\beta_{K,K})\beta_{\Gamma,L}}{1-s+\beta_{K,L}}] - (s-\beta_{L,L})[1+\alpha_{\Gamma,L} - \theta\alpha_{L,K}\alpha_{K,L}] \\ & -(1+\alpha_{L,L})\beta_{\Gamma,L} \end{aligned}$$

Within each case, due to geometrical arguments, we have several configurations: 4 within the first case and 2 within the second case. These configurations are presented below and depicted in the plane (T, D) in Figure 1.

- Configuration (i) $D_1'(\sigma) < 0$ and $S_1 \in (0,1)$: $I_1 < -\theta I_2 < 0$.
- Configuration (ii): $D_1'(\sigma) < 0$ and $|S_1| > 1$:

  either $I_1 < 0$ and $I_2 < 0$, where $S_1 > 1$ or $0 < 2I_1 < -\theta I_2$, where $S_1 < -1$;
- Configuration (iii): $D_1'(\sigma) < 0$ and $S_1 \in (-1, S_B)$: $0 < (1-S_B)\, I_1 < -\theta I_2 < 2I_1$;
- Configuration (iv): $D_1'(\sigma) < 0$ and $S_1 \in (S_B, 0)$: $0 < I_1 < -\theta I_2 < (1-S_B)\, I_1$;
- Configuration (v): $D_1'(\sigma) > 0$ and $S_1 \in (0, S_D)$: $-\frac{I_1}{\theta} < I_2 < -\frac{(1+\alpha_{LL})(1-s)}{s-\beta_{LL}} I_1 < 0$;
- Configuration (vi): $D_1'(\sigma) > 0$ and $S_1 \in (S_D, 1)$: $-\frac{(1+\alpha_{LL})(1-s)}{s-\beta_{LL}} I_1 < I_2 < 0$.

where $S_D \in (0,1)$ and $S_B \in (-1,0)$, the critical value of S_1 such that the Δ_1 line goes through B, are given in the Appendix 6.7.

Before proceeding let us make a few helpful remarks. When $\alpha_{\Gamma,i} = \beta_{\Gamma,i} = 0$, using Assumption 4, $I_1 + \theta I_2$ becomes:

$$I_1 + \theta I_2 = \theta(1+\alpha_{L,L})(1-s-\beta_{K,K}) - (s-\beta_{L,L})(1+\theta\alpha_{K,K}), \tag{19}$$

which is always strictly negative under Assumption 1.2. Hence $D_1'(\sigma) < 0$.

When $\alpha_{K,i} = \beta_{K,i} = 0$, $I_1 + \theta I_2$ becomes:

$$\begin{aligned} I_1 + \theta I_2 = & - (s - \beta_{L,L})(1 + \alpha_{\Gamma,L}) - \beta_{\Gamma,L}(1 + \alpha_{L,L}) \\ & + \theta\,(1 - s)\,(1 + \alpha_{L,L})\,(1 + \alpha_{\Gamma,L} + \alpha_{\Gamma,K})\,, \end{aligned} \tag{20}$$

which a priori can take a positive or negative sign. Using (20) and (18), configurations (v) and (vi), where $D_1'(\sigma) > 0$ and $\alpha_{K,i} = \beta_{K,i} = 0$, are obtained when

$$\alpha_{\Gamma L} < \alpha_{\Gamma L}^* \equiv -1 - \frac{\beta_{\Gamma,L}(1 + \alpha_{L,L}) - \theta\,(1 - s)\,(1 + \alpha_{L,L})\,\alpha_{\Gamma,K}}{(s - \beta_{L,L}) - \theta\,(1 - s)\,(1 + \alpha_{L,L})}. \tag{21}$$

Finally, note that, using (11) and (12), when $\alpha_{K,i} = \beta_{K,i} = 0$, the condition $D > T - 1$ can be written as:

$$\theta\,(1 - s)\,\frac{[(\alpha_{L,L} + \alpha_{L,K}) - (\alpha_{\Gamma,L} + \alpha_{\Gamma,K})] - (\epsilon_\gamma - 1)\,\sigma}{\sigma(1 + \alpha_{L,L}) - (s - \beta_{L,L})} > 0 \tag{22}$$

Since $\epsilon_\gamma - 1 > 0$, this condition can only be satisfied, under Assumption 1, when $(\alpha_{L,L} + \alpha_{L,K}) - (\alpha_{\Gamma,L} + \alpha_{\Gamma,K}) > 0$. This last condition implies that, when $\alpha_{K,i} = \beta_{K,i} = 0$, $I_2 \equiv -\,(1 + \alpha_{LL})\,(1{-}s)\,[(\alpha_{L,L} + \alpha_{L,K}) - (\alpha_{\Gamma,L} + \alpha_{\Gamma,K})] < 0$. Therefore $D_1'(\sigma) > 0 \Leftrightarrow I_1 + \theta I_2 > 0$, requires $I_1 > 0$ and $0 < S_1 < 1$.

6.4.1 Derivation of Proposition 1

The case where $D_1'(\sigma) < 0$: Proposition 1 (a) - Table 1

Configuration (i) **($S_1 \in (0, 1)$)** In this configuration, the half line Δ_1 starts (for $\sigma = +\infty$) on the line (AC) between A and C (see Assumptions 2 and 4), with a slope lower than 1, i.e., lower than the slope of (AC), and points upwards, lying therefore on the right of (AC). See Figure 2. Two main cases can arise. If $\alpha_{KK} \leq 0$, then $S < 1$ (see Lemma 1) and the half-line Δ lies below line (AC) and above line (AB). Hence, the steady state is a saddle.

If $\alpha_{KK} > 0$, there exists the critical value σ_T such that $S = 1$ (see Lemma 1). Hence, for $\sigma \leq \sigma_T$, since $S \leq 1$ the same as before happens. However, if $\sigma > \sigma_T$, then $S > 1$, and the half-line Δ will cross (AC). We define σ_{H_2} as a critical value of σ such that the half-line Δ goes through point C. See Appendix 6.6. Then, for $\sigma = +\infty$, the half-line Δ starting on (AC) points upwards with a slope higher than 1. By continuity, the critical value σ_{H_2}, in this configuration greater than σ_T, exists.[60] For $\sigma > \sigma_{H_2}$, Δ crosses $[BC]$ after

[60] In Appendix 6.8 we show the uniqueness of σ_{H_2} in the configuration under analysis.

crossing (AC), i.e., the steady state is a saddle for $1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$, undergoes a transcritical bifurcation at $\varepsilon_\gamma = \varepsilon_{\gamma_T}$, becomes a sink for $\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$, undergoes a Hopf bifurcation at $\varepsilon_\gamma = \varepsilon_{\gamma_H}$, and becomes a source for $\varepsilon_\gamma > \varepsilon_{\gamma_H}$. For $\sigma_T < \sigma < \sigma_{H_2}$, the Hopf bifurcation disappears and the steady state is either a saddle ($1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$) or a source ($\varepsilon_\gamma > \varepsilon_{\gamma_T}$).

Configuration *(ii)* **($|S_1| > 1$)** Since the half-line Δ_1 starts on the line (AC), between A and C, and points upwards with a slope S_1 strictly greater than 1 or strictly smaller than -1, it crosses neither (AB), nor (AC). See Figure 3. However, since Δ_1 crosses the segment $[BC]$, we define σ_{H_1} as the critical value of σ such that $D_1(\sigma_{H_1}) = 1$. See (26).

As in the previous configuration, the analysis depends on the value of $\alpha_{K,K}$. Consider first that $\alpha_{K,K} \leq 0$, which means that $S < 1$. See Lemma 1. If $\sigma \leq \sigma_{H_1}$, the half-line Δ starts above $[BC]$ and crosses (AC). For $\sigma > \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is inside (ABC). When $\sigma_{H_1} < \sigma < \sigma_{H_2}$, Δ crosses first the segment $[BC]$ and then line (AC), above point C. For $\sigma > \sigma_{H_2}$, Δ only crosses (AC) below point C.[61]

Assuming now that $\alpha_{K,K} > 0$, the critical value $\sigma_T > 0$ exists (see Lemma 1). We assume that $\sigma_T > \sigma_{H_1}$, i.e., the slope of the half-line Δ at σ_{H_1} is lower than 1. This is ensured by:

Assumption 5 *If $\alpha_{K,K} > 0$, then $-1 - T_1(\sigma_T) < D_1(\sigma_T) < 1$.*[62].

As before, when $\sigma \leq \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is above the segment $[BC]$ and the half-line Δ only crosses (AC). To simplify the analysis, we also assume that:

Assumption 6 *If $\sigma > \sigma_{H_1}$ and $\alpha_{K,K} > 0$, then $\varepsilon_{\gamma_H} < \varepsilon_{\gamma_T}$.*[63]

As a consequence, when $\sigma_{H_1} < \sigma < \sigma_T$, Δ crosses first the segment $[BC]$ and then line (AC) above C. When $\sigma \geq \sigma_T$, the half-line Δ only crosses $[BC]$.

Configuration *(iii)* **($S_1 \in (-1, S_B)$)** In this configuration the slope S_1 is negative and greater than -1, and the half-line Δ_1, that starts on (AC) and points upwards to the left, crosses (AB) above point B. See Figure 4. Let us then define σ_F as the critical value such that $1 + D_1(\sigma_F) + T_1(\sigma_F) = 0$. See (27).

[61] This last case does not appear if σ_{H_2} does not exist. See Appendix 6.8, for $\alpha_{K,K} = 0$.

[62] More precisely, Assumption 5 ensures that, for configurations *(ii)*, *(iii)* and *(iv)*, the point $(T_1(\sigma_T), D_1(\sigma_T))$ lies within the triangle (ABC) when $\alpha_{KK} > 0$.

[63] This condition is ensured if (31) in Appendix 6.8 is satisfied.

Consider first that $\alpha_{K,K} \leq 0$, i.e., the slope of Δ is always smaller or equal to 1. When $\sigma < \sigma_F$, $(T_1(\sigma), D_1(\sigma))$ is below line (AB) and above segment $[BC]$. Since the half-line Δ points upwards it does not cross $[BC]$, but crosses (AB) before crossing (AC). When $\sigma_F \leq \sigma \leq \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is above (AB) and above $[BC]$. Then, Δ only crosses (AC). When $\sigma > \sigma_{H_1}$, the point $(T_1(\sigma), D_1(\sigma))$ is inside the triangle (ABC). As in the previous configuration, σ_{H_2} is greater than σ_{H_1}. Therefore, for $\sigma_{H_1} \leq \sigma < \sigma_{H_2}$, the half-line Δ crosses first $[BC]$ and then (AC) above C, and for $\sigma > \sigma_{H_2}$, the half-line Δ only crosses (AC) below C.[64]

Consider now that $\alpha_{K,K} > 0$. In this case, the critical value $\sigma_T > 0$ exists and, under Assumption 5, we have $\sigma_T > \sigma_{H_1} > \sigma_F$. Therefore, when $\sigma < \sigma_{H_1}$, we obtain the same results as before. When $\sigma_{H_1} < \sigma < \sigma_T$, $(T_1(\sigma), D_1(\sigma))$ is inside the triangle (ABC) and, under Assumption 6, Δ crosses first $[BC]$, and then (AC) above point C. When $\sigma \geq \sigma_T$, S becomes greater than 1, which means that Δ only crosses $[BC]$.

Configuration (iv) **($S_1 \in (S_B, 0)$)** In this configuration, the slope S_1 is negative and greater than -1, and the half line Δ_1, that points upwards to the left, crosses line (AB) below point B. See Figure 5. In this configuration, a new critical value, σ_{H_3}, the value of σ such that the half line Δ goes through point B, becomes relevant.[65]

We begin by assuming $\alpha_{K,K} \leq 0$, which implies S smaller than 1. For $\sigma < \sigma_{H_3}$, Δ starts on the left-side of (AB), crosses (AB) above B and (AC). For $\sigma_{H_3} < \sigma < \sigma_F$, Δ also starts on the left-side of (AB), but crosses (AB) below B, the segment $[BC]$, and (AC) above C. Recall that when σ_{H_2} exists, the Δ line crosses point C. Then, for $\sigma_F \leq \sigma < \sigma_{H_2}$,[66] $(T_1(\sigma), D_1(\sigma))$ is inside (ABC), and Δ crosses $[BC]$ and (AC) above C. For $\sigma > \sigma_{H_2}$, $(T_1(\sigma), D_1(\sigma))$ is still inside (ABC) and Δ crosses (AC) below C.[67]

We consider now the case where $\alpha_{K,K} > 0$. Since we assume that σ_T is sufficiently big, we have that $\sigma_T > \sigma_F$ $(> \sigma_{H_3})$ (Assumption 5). Then, for $\sigma < \sigma_{H_3}$, the half-line Δ crosses (AB) above B and (AC). For $\sigma_{H_3} < \sigma < \sigma_F$, Δ crosses (AB) below B, the segment $[BC]$ and (AC) above C (Assumption 6). For $\sigma_F \leq \sigma < \sigma_T$, Δ starts inside (ABC) with a slope smaller than 1.

[64] This last case does not appear if σ_{H_2} does not exist. See Appendix 6.8, for $\alpha_{K,K} = 0$.

[65] In Appendix 6.9 we prove that in this configuration, there exists a unique critical value $\sigma_{H_3} \in (\sigma_{H_1}, \sigma_F)$ such that the half-line Δ goes through point B and crosses $[BC]$ on the right of B for $\sigma > \sigma_{H_3}$.

[66] Note that σ_{H_2} may be higher or lower than σ_F. To simplify the exposition we only present in Table 1 the results for $\sigma_{H_2} > \sigma_F$. Using geometrical considerations, the reader can easily deduce the case where $\sigma_F > \sigma_{H_2}$.

[67] This last case does not appear if σ_{H_2} does not exist. See Appendix 6.8, for $\alpha_{K,K} = 0$.

Then, it crosses $[BC]$ and (AC) above C. For $\sigma \geq \sigma_T$, the slope S being greater than 1, Δ only crosses $[BC]$.

All these results are summarized in Table 1.

The case where $D_1'(\sigma) > 0$ and $\alpha_{K,j} = \beta_{K,j} = 0$, $j = K, L$: Proposition 1 (b) - Table 2 In this case the half-line Δ_1 points downwards and, since $S_1 \in (0,1)$, it lyes on the left of (AC). Remember that, from Assumptions 2, and 4 the starting point of Δ_1, $(T_1(+\infty), D_1(+\infty))$, is on the (AC) line, between A and C. Therefore Δ_1 crosses (AB) between A and B at the critical value $\sigma_F \in (\frac{s-\beta_{L,L}}{1+\alpha_{L,L}}, +\infty)$.

The half-line Δ, beginning on line Δ_1 for $\varepsilon_\gamma = 1$, points upwards. Since $D_1'(\sigma) > 0$, it must then cross (AB) at $\varepsilon_{\gamma_F} > 1$ if and only if $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_F$. Moreover, Δ always crosses (AC) at a value $\varepsilon_{\gamma_T} > 1$, since from Lemma 1, it has a slope $S \in (0,1)$, with S tending to 1 when σ tends to $+\infty$. Notice also that, since $D_1(+\infty) < 1$ and $D_1'(\sigma) > 0$, the half-line Δ, pointing upwards, also always crosses the line (BC), defined by $D = 1$, at $\varepsilon_{\gamma H} > 1$. However, whether Hopf bifurcations occur or not, depend on whether Δ crosses the segment $[BC]$ in its interior or not. The following Lemma will help us with this question:

Lemma 2 *Let* $S_D \equiv 1 - \frac{\theta(1+\alpha_{LL})(1-s)}{s-\beta_{LL}}$.

1. *If* $S_1 < S_D$, *(i) when* $\alpha_{\Gamma,K} \leq \alpha_{L,K}$ *then* $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$; *(ii) when* $\alpha_{\Gamma,K} > \alpha_{L,K}$, *then* $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ *for* $\frac{s-\beta_{L,L}}{1+\alpha_{L,L}} < \sigma < \sigma_{H_2}$ *and* $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ *for* $\sigma > \sigma_{H_2}$.

2. *If* $S_1 > S_D$, *(i) when* $\alpha_{\Gamma,K} \geq \alpha_{L,K}$ *then* $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$; *(ii) when* $\alpha_{\Gamma,K} < \alpha_{L,K}$, *then:* $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ *for* $\frac{s-\beta_{L,L}}{1+\alpha_{L,L}} < \sigma < \sigma_{H_2}$ *and* $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ *for* $\sigma > \sigma_{H_2}$.

Proof. With $\alpha_{K,i} = \beta_{K,i} = 0$, $i = K, L$, note that $\epsilon_{\gamma_H} > \epsilon_{\gamma_T} \Leftrightarrow (\sigma - \sigma_{H_2})(\alpha_{\Gamma,K} - \alpha_{L,K}) > 0$, where $\sigma_{H_2} \equiv \frac{s-\beta_{L,L}}{1+\alpha_{L,L}} - \frac{\frac{s-\beta_{L,L}}{(1+\alpha_{L,L})(1-s)}\frac{I_4-I_3}{\theta}(S_1-S_D)}{(\alpha_{\Gamma,K}-\alpha_{L,K})(1+\alpha_{L,L})}$ is the value of σ such that $\epsilon_{\gamma_H}(\sigma_{H_2}) = \epsilon_{\gamma_T}(\sigma_{H_2})$. Hence, when $S_1 < S_D$ and $\alpha_{\Gamma,K} < \alpha_{L,K}$ or when $S_1 > S_D$ and $\alpha_{\Gamma,K} > \alpha_{L,K}$, we have $\sigma_{H_2} \leq \frac{s-\beta_{L,L}}{1+\alpha_{L,L}}$ so that $\sigma > \sigma_{H_2}$ for all σ under consideration. Otherwise, we get $\sigma_{H_2} > \frac{s-\beta_{L,L}}{1+\alpha_{L,L}}$. In the case where $\alpha_{\Gamma,K} = \alpha_{L,K}$, $(\sigma - \sigma_{H_2})(\alpha_{\Gamma,K} - \alpha_{L,K}) = \frac{s-\beta_{L,L}}{(1+\alpha_{L,L})^2(1-s)}\frac{I_4-I_3}{\theta}(S_1 - S_D)$. Therefore, $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $S_1 > S_D$ and $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $S_1 < S_D$. Lemma 2 immediately follows.

According to this Lemma, it is convenient to analyze the local dynamics considering separately configuration (v) where $S_1 \in (0, S_D)$ and configuration (vi) where $S_1 \in (S_D, 1)$.

Before proceeding, let us also establish the relevant result:

Lemma 3 *For $D_1'(\sigma) > 0$ with $\alpha_{K,j} = \beta_{K,j} = 0$, and under Assumption 1 , if $0 < S_1 < S_D$, then $S > S_1$ for all $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$; while if $S_D < S_1 < 1$, then $S < S_1 \Leftrightarrow \frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma^{S_1}$, where σ^{S_1} is given by (29).*

Proof. Notice that using (13), (16), Lemma 2 and $\alpha_{K,j} = \beta_{K,j} = 0$, since $D_1'(\sigma) > 0$, we can write $S_1 - S_D = -\frac{\theta I_2}{I_1}\frac{1+\alpha_{LL}}{s-\beta_{LL}}\left[\sigma^{S_1} - \frac{s-\beta_{LL}}{1+\alpha_{LL}}\right]$ and $S - S_1 = -\frac{\theta I_2}{\sigma I_1}\left[\sigma - \sigma^{S_1}\right]$, where $I_2 < 0$ and $I_1 > 0$, and $\frac{1+\alpha_{LL}}{s-\beta_{LL}} > 0$ under Assumption 1. This implies $S > S_1 \Leftrightarrow \sigma > \sigma^{S_1}$, while $\sigma^{S_1} < \frac{s-\beta_{LL}}{1+\alpha_{LL}} \Leftrightarrow S_1 < S_D$. Therefore, when $S_1 < S_D$, $S > S_1$ for all $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$. When $S_1 > S_D$ then $S < S_1 \Leftrightarrow \frac{s-\beta_{LL}}{(1+\alpha_{LL})} < \sigma < \sigma^{S_1}$.

Configuration (v) **$(S_1 \in (0, S_D))$** Using Lemma 3, we show that in this configuration $S_1 < S$. Hence, for $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_F$, the half-line Δ crosses (AC) only after having crossed line (AB), i.e. $\varepsilon_{\gamma_T} > \varepsilon_{\gamma F} > 1$ (see Figure 6). For $\sigma < \sigma_{H_3}$, Δ crosses line (BC) on the left of point B, i.e., $\varepsilon_{\gamma_H} < \varepsilon_{\gamma F}$, whereas for $\sigma > \sigma_{H_3}$ the crossing point lies on the right of point B, i.e., $\varepsilon_{\gamma H} > \varepsilon_{\gamma_F}$.[68] In the first case ($\sigma < \sigma_{H_3}$) there are no Hopf bifurcations. However, for $\sigma > \sigma_{H_3}$, Hopf bifurcations are possible if the crossing point lies on the left of point C, i.e., if $\varepsilon_{\gamma_H} < \varepsilon_{\gamma_T}$. Under Lemma 2, this will depend on the sign of $\alpha_{\Gamma,K} - \alpha_{L,K}$ and also on whether σ is higher or lower than σ_{H_2}. With the help of geometrical arguments we can see that when $\sigma_{H_2} > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$ exists, then $\sigma_{H_2} > \sigma_{H_3}$.[69] However σ_{H_2} may be higher or lower than σ_F. To simplify the exposition we only present in Table 2 the results for this configuration assuming that $\sigma_{H_2} > \sigma_F$.[70]

Configuration (vi) **$(S_1 \in (S_D, 1))$** In this configuration, as shown in Lemma 3, $S < S_1$ for all $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma_{S_1}$ and $S > S_1$ for all $\sigma > \sigma_{S_1}$, where σ_{S_1} is the value of σ for which $S = S_1$.[71] Define σ_{S_2} as the critical value of σ such that the half line Δ goes through point A.[72] When $\frac{s-\beta_{LL}}{1+\alpha_{LL}} <$

[68] We can see geometrically that $\sigma_{H_3} < \sigma_F$.

[69] Suppose on the contrary that $\sigma_{H_2} < \sigma_{H_3}$. For $\sigma_{H_2} < \sigma < \sigma_{H_3}$, Δ could not cross the line (BC) on the right of point C because for $\sigma < \sigma_{H_3}$, as shown in the Appendix 6.9, it must cross line (BC) on the left of point B.

[70] Using geometrical considerations, the reader can easily obtain the case where $\sigma_F > \sigma_{H_2}$.

[71] The expression for σ_{S_1} is given in (29).

[72] The expression for σ_{S_2} is given in (30). As the slope of Δ increases and its initial point shifts upwards along Δ_1, $\varepsilon_{\gamma T} < \varepsilon_{\gamma F}$ for $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma < \sigma^{S_2}$, and $\varepsilon_{\gamma T} > \varepsilon_{\gamma F}$

$\sigma < \sigma_{S_2}$, the half-line Δ, pointing upwards with a slope lower than Δ_1 (we can see geometrically that $\sigma_{S_2} < \sigma_{S_1}$), crosses first (AC) and then (AB), both crossing points being below point A. See Figure 7. What happens for $\sigma > \sigma_{S_2}$ depends on whether Δ crosses (AB), which will only happen for $\sigma < \sigma_F$, and on whether Δ crosses the segment $[BC]$ in its interior or not. We will assume that (T_1, D_1) for $\sigma = \sigma_{S_1}$ is inside the triangle (ABC), i.e., that:

Assumption 7 $D_1(\sigma_{S_1}) > -1 - T_1(\sigma_{S_1})$

Hence, $\sigma_{S_1} > \sigma_F$.[73] Then, for $\sigma_{S_2} < \sigma < \sigma_F$, Δ still has a slope lower than Δ_1, but it first crosses (AB) and then (AC). For $\sigma > \sigma_F$, Δ no longer crosses (AB). Whether Δ goes through (BC) on the left or on the right of point C depends, according to Lemma 3, on the sign of $\alpha_{\Gamma K} - \alpha_{LK}$ and on whether σ is higher or lower than σ_{H_2}. We can see geometrically that $\sigma_{H_2} > \sigma_{S_1}$ and therefore $\sigma_{H_2} > \sigma_F$.

All these results are summarized in Table 2.

6.5 Expressions for critical values of ε_γ

ε_{γ_H} is such that $D = 1$, which is equivalent to:

$$\begin{aligned}\varepsilon_{\gamma_H} &= 1 + \{\sigma[\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,K}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{\Gamma,L}))] \\ &+ \theta[(1-s-\beta_{K,K})(1+\alpha_{\Gamma,L}) + \alpha_{\Gamma,K}(1-s+\beta_{K,L}) \\ &- \alpha_{K,L}\beta_{\Gamma,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}] - \beta_{\Gamma,L}(1+\theta\alpha_{K,K}) - (s-\beta_{L,L})\} \\ &/[\sigma(1+\theta\alpha_{K,K}) - \theta(1-s-\beta_{K,K})]\end{aligned} \tag{23}$$

ε_{γ_F} is such that $1 + T + D = 0$. After some computations, we obtain:

for $\sigma > \sigma^{S_2}$. Easy analytical computations show that $\sigma_{S_2} \in \left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \infty\right)$ exists if $\frac{s-\beta_{LL}}{1+\alpha_{LL}}(1+\alpha_{LL}+\alpha_{LK}-\alpha_{\Gamma K}) > \theta(1-s)(1+\alpha_{LL}+\alpha_{LK}) - \beta_{\Gamma L}$. Hence, if this condition is not met, then $\varepsilon_{\gamma T} > \varepsilon_{\gamma F}$ for all $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$.

[73] Note that then we have $\varepsilon_{\gamma_H} > \varepsilon_{\gamma_F}$ for all $\sigma > \frac{s-\beta_{LL}}{(1+\alpha_{LL})}$. See Appendix 6.9 for the definition and the existence of σ_{H_3}. Indeed, there cannot exist a $\sigma_{H_3} > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$, since its existence would require that $\sigma_{S_1} < \sigma_{H_3} < \sigma_F$, which is ruled out by Assumption 7.

$$
\begin{aligned}
\varepsilon_{\gamma_F} = 1 + \{&\sigma[2(2+\alpha_{L,L}+\alpha_{\Gamma,L}) + \theta(\alpha_{K,K}(2+\alpha_{L,L}+\alpha_{\Gamma,L}) - \alpha_{K,L}(\alpha_{L,K}+\alpha_{\Gamma,K}))] \\
&- 2(s-\beta_{L,L}-\beta_{\Gamma,L}) - \theta[(1-s-\beta_{K,K})(2+\alpha_{L,L}+\alpha_{\Gamma,L}) \\
&+ \alpha_{K,K}(s-\beta_{L,L}-\beta_{\Gamma,L}) + (\alpha_{L,K}+\alpha_{\Gamma,K})(1-s+\beta_{K,L}) \\
&+ \alpha_{K,L}(s-\beta_{L,L}-\beta_{\Gamma,L})\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}]\}/[\theta(1-s-\beta_{K,K}) - \sigma(2+\theta\alpha_{K,K})]
\end{aligned}
\tag{24}
$$

ε_{γ_T} is such that $1-T+D=0$. After some computations, we obtain:

$$
\begin{aligned}
\varepsilon_{\gamma_T} = 1 + \{&(1-s-\beta_{K,K})(\alpha_{L,L}-\alpha_{\Gamma,L}) + \alpha_{K,K}(s-\beta_{L,L}+\beta_{\Gamma,L}) + \\
&+ (\alpha_{L,K}-\alpha_{\Gamma,K})(1-s+\beta_{K,L}) + \alpha_{K,L}(s-\beta_{L,L}+\beta_{\Gamma,L})\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}\} \\
&/(1-s-\beta_{K,K}-\sigma\alpha_{K,K})
\end{aligned}
\tag{25}
$$

6.6 Definitions and expressions for some critical values of σ

σ_{H_1} is the critical value of σ such that $D_1(\sigma_{H_1}) = 1$.

$$
\begin{aligned}
\sigma_{H_1} \equiv &\frac{s-\beta_{L,L}+\beta_{\Gamma,L}(1+\theta\alpha_{K,K})}{\alpha_{L,L}-\alpha_{\Gamma,L}-\theta[\alpha_{K,K}(1+\alpha_{\Gamma,L})-\alpha_{\Gamma,K}\alpha_{K,L}]} \\
&-\frac{\theta[(1-s-\beta_{K,K})(1+\alpha_{\Gamma,L}) + \alpha_{\Gamma,K}(1-s+\beta_{K,L}) - \alpha_{K,L}\beta_{\Gamma,L}\frac{1-s-\beta_{K,K}}{1-s+\beta_{K,L}}]}{\alpha_{L,L}-\alpha_{\Gamma,L}-\theta[\alpha_{K,K}(1+\alpha_{\Gamma,L})-\alpha_{\Gamma,K}\alpha_{K,L}]}
\end{aligned}
\tag{26}
$$

σ_{H_2} is a critical value of σ such that the half-line Δ goes through the point $(T, D) = (2, 1)$, i.e., goes through point C.[74] Note that $\varepsilon_{\gamma_T} = \varepsilon_{\gamma_H}$ for $\sigma = \sigma_{H_2}$.

σ_{H_3} is the critical value of σ such that the half line Δ goes through the point $(T, D) = (-2, 1)$, i.e., goes through point B.[75] Note that $\varepsilon_{\gamma_F} = \varepsilon_{\gamma_H}$ for $\sigma = \sigma_{H_3}$.

[74] In Appendix 6.8, we show conditions for its existence and uniqueness.

[75] In Appendix 6.9, we we show conditions for its existence and uniqueness.

The critical value σ_F is defined by $1 + D_1(\sigma_F) + T_1(\sigma_F) = 0$.[76]

$$\sigma_F \equiv \frac{(s - \beta_{LL} - \beta_{\Gamma,L})\left[2 + \theta(\alpha_{KK} + \alpha_{KL}\frac{1-s-\beta_{KK}}{1-s+\beta_{KL}}\right]}{(2 + \theta\alpha_{KK})(2 + \alpha_{LL} + \alpha_{\Gamma,L}) - \theta\alpha_{KL}(\alpha_{LK} + \alpha_{\Gamma,K})} + \frac{\theta[(1 - s - \beta_{KK})(2 + \alpha_{LL} + \alpha_{\Gamma,L}) + (1 - s + \beta_{KL})(\alpha_{LK} + \alpha_{\Gamma,K})]}{(2 + \theta\alpha_{KK})(2 + \alpha_{LL} + \alpha_{\Gamma,L}) - \theta\alpha_{KL}(\alpha_{LK} + \alpha_{\Gamma,K})} \tag{27}$$

σ_T is the value of σ for which $S = 1$.

$$\sigma_T \equiv \frac{(1 - s - \beta_{KK})}{\alpha_{KK}} \tag{28}$$

σ_{S_1} is the value of σ for which $S = S_1$, relevant for Lemma . For $\alpha_{Ki} = \beta_{Ki} = 0$ it is given by:

$$\sigma^{S_1} \equiv \frac{\theta\,(1 - s)\,(1 + \alpha_{LL})\,(1 + \alpha_{LL} + \alpha_{LK}) - (s - \beta_{LL})\,(1 + \alpha_{\Gamma,L}) - (1 + \alpha_{LL})\,\beta_{\Gamma L}}{(1 + \alpha_{LL})\,(\alpha_{LL} + \alpha_{LK} - \alpha_{\Gamma,L} - \alpha_{\Gamma K})} \tag{29}$$

σ_{S_2} is the value of σ such that the half line Δ goes through the point $(T, D) = (-1, 0)$, i.e., goes through point A, so that $\varepsilon_{\gamma T} = \varepsilon_{\gamma_F}$. For $\alpha_{Ki} = \beta_{Ki} = 0$ it is given by:

$$\sigma_{S_2} \equiv \frac{\theta\,(1 - s)\,(1 + \alpha_{LL} + \alpha_{LK}) + (s - \beta_{LL} - \beta_{\Gamma L})}{2\,(1 + \alpha_{LL}) + \alpha_{LK} - \alpha_{\Gamma K}} \tag{30}$$

6.7 Expressions for critical values of S_1

$S_B < 0$ is the critical value of S_1 such that the Δ_1 line goes through B and is given by:

$$S_B = 1 + \frac{4(1 + \alpha_{L,L})}{-3(1 + \alpha_{L,L}) - (1 + \theta\alpha_{K,K})(1 + \alpha_{\Gamma,L}) + \theta\alpha_{\Gamma,K}\alpha_{K,L}}$$

$$S_D \equiv 1 - \frac{\theta(1 + \alpha_{LL})(1 - s)}{s - \beta_{LL}}$$

[76] Note that using Assumption 4 the denominator of σ_F can also be written as $2(2 + \alpha_{L,L} + \alpha_{\Gamma,L}) - 2\theta[\alpha_{\Gamma,K}\alpha_{K,L} - \alpha_{K,K}(1 + \alpha_{\Gamma,L}) > 0$

6.8 Existence of σ_{H_2}

Recall that when $\sigma = \sigma_{H_2}$ we have $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$, i.e. the Δ line goes through point C. To discuss the existence and uniqueness of σ_{H_2}, we consider first the configurations where $S_1 \in (0,1)$, and then the remaining ones.

1. Configurations where $S_1 \in (0,1)$.

When $D_1'(\sigma) < 0$ (as in configuration (i)), the existence of σ_{H_2} requires $\alpha_{K,K} > 0$. Since $D_1'(\sigma) < 0$ and $S(\sigma)$ increases with σ (with $S(+\infty) > 1$), we deduce by direct geometrical considerations the existence and uniqueness of $\sigma_{H_2}(> \sigma_T)$, such that $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $\sigma < \sigma_{H_2}$, and $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $\sigma > \sigma_{H_2}$.

When $D_1^{'}(\sigma) > 0$, see Lemma 2.

2. Configurations where $S_1 > 1$ **or** $S_1 < 0$.

Consider first the case where $\alpha_{K,K} < 0$. Note that the equation $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$ is a polynomial of degree 2, i.e. has at most two solutions. Since $S(+\infty) \in (0,1)$, we can see geometrically that a solution $\sigma_{H_2} \in (\sigma_{H_1}, +\infty)$ must exist and the number of these solutions is odd. We deduce the uniqueness of $\sigma_{H_2}(> \sigma_{H_1})$, such that $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $\sigma < \sigma_{H_2}$, and $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $\sigma > \sigma_{H_2}$.

Consider now that $\alpha_{K,K} = 0$. Note that in this particular case, ϵ_{γ_T} does not depend on σ. The equation $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$ has at most one solution $\sigma_{H_2} \in (\sigma_{H_1}, +\infty)$, and this solution is by continuity such that again $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for $\sigma < \sigma_{H_2}$, and $\epsilon_{\gamma_H} > \epsilon_{\gamma_T}$ for $\sigma > \sigma_{H_2}$.

Finally, consider that $\alpha_{K,K} > 0$. We can see geometrically that if there is a solution σ_{H_2} to $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$, then $\sigma_{H_2} \in (\sigma_{H_1}, \sigma_T)$. The inequality $\epsilon_{\gamma_H} \leq \epsilon_{\gamma_T}$ is equivalent to $g(\sigma) \geq 0$, where

$$\begin{aligned} g(\sigma) \equiv \alpha_{K,K}[\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,L}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{\Gamma,L}))](\sigma - \sigma_T)(\sigma - \sigma_{H_1}) \\ - \frac{I_2}{1+\alpha_{L,L}}[\sigma(1+\theta\alpha_{K,K}) - \theta(1 - s - \beta_{K,K})] \end{aligned}$$

This function describes a convex parabola with $g(0) > 0$, $g(\sigma_{H_1}) > 0$, $g(\sigma_T) > 0$ and $g(+\infty) = +\infty$. Hence, either we have two solutions (if $g'(\sigma_{H_1}) < 0$) or none (if $g'(\sigma_{H_1}) \geq 0$) to the equation $g(\sigma) = 0$. As:

$$\begin{aligned} g'(\sigma) = \alpha_{K,K}[\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,L}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{\Gamma,L}))] \\ [2\sigma - (\sigma_T + \sigma_{H_1})] - \frac{I_2}{1+\alpha_{L,L}}(1+\theta\alpha_{K,K}) \end{aligned}$$

We deduce that $g'(\sigma_{H_1}) \geq 0$ is equivalent to:

$$\begin{aligned} I_2 \leq & \alpha_{K,K}(1+\alpha_{L,L})[\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,L}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{\Gamma,L}))] \\ & (\sigma_{H_1} - \sigma_T)/(1+\theta\alpha_{K,K}) \end{aligned} \tag{31}$$

Hence, when this inequality is satisfied, there is no solution to $g(\sigma) = 0$, because $g(\sigma_T) \geq g(\sigma_{H_1}) > 0$. This implies that $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for all $\sigma > \sigma_{H_1}$, i.e. the half-line Δ always goes above point C.

6.9 Existence of σ_{H_3}

Using (23) and (24) we have that $\epsilon_{\gamma H} = \epsilon_{\gamma F} \Leftrightarrow h(\sigma) = 0$, and $\epsilon_{\gamma H} > \epsilon_{\gamma F} \Leftrightarrow h(\sigma) > 0$, where:

$$\begin{aligned}h(\sigma) \equiv &[\sigma(2+\theta\alpha_{K,K}) - \theta(1-s-\beta_{K,K})][\alpha_{L,L} - \alpha_{\Gamma,L} + \theta(\alpha_{\Gamma,K}\alpha_{K,L} \\ &- \alpha_{K,K}(1+\alpha_{\Gamma,L}))](\sigma - \sigma_{H_1}) + 2[\sigma(1+\theta\alpha_{K,K}) - \theta(1-s-\beta_{K,K})] \\ &[2 + \alpha_{L,L} + \alpha_{\Gamma,L} - \theta(\alpha_{\Gamma,K}\alpha_{K,L} - \alpha_{K,K}(1+\alpha_{L,L}))](\sigma - \sigma_F),\end{aligned}$$

By definition, σ_{H_3} is a value of σ such that $\epsilon_{\gamma H} = \epsilon_{\gamma F}$, therefore it must be a solution of $h(\sigma) = 0$.

Since $h(\sigma)$ is a polynomial of degree 2, the equation $h(\sigma) = 0$ has at most two solutions. Here we limit our analysis to configurations (iv), (v) and (vi) since σ_{H_3} is only relevant under these configurations. In all of them, since Δ is positively sloped pointing upwards, it can only go through point B if its initial point in Δ_1 is on the left of line (AB), i.e., $\sigma_{H_3} < \sigma_F$. Also, in all these three configurations the polynomial $h(\sigma)$ is a convex function of σ since the coefficient of the quadratic term σ^2 is positive.[77]

Consider first configuration (iv). We can see geometrically that if there is a $\sigma_{H_3} > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$ then it must satisfy $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma_{H_1} < \sigma_{H_3} < \sigma_F$. Straight computations show that in this configuration $h(\sigma_F) > 0$ and $h(\sigma_{H_1}) < 0$. Therefore there is a unique $\sigma_{H_3} \in (\sigma_{H_1}, \sigma_F)$ such that $h(\sigma_{H_3}) = 0$. By continuity, we have that $\epsilon_{\gamma H} > \epsilon_{\gamma F}$ for $\sigma_F > \sigma > \sigma_{H_3}$, and $\epsilon_{\gamma H} < \epsilon_{\gamma F}$ for $\sigma_{H_1} < \sigma < \sigma_{H_3}$.

Consider now configurations (v) and (vi). As seen above if $\sigma_{H_3}(> \frac{s-\beta_{LL}}{1+\alpha_{LL}})$ exists it must satisfy $\sigma_{H_3} < \sigma_F$. Straight computations show that in these configurations $h(\sigma_F) > 0$. In configuration (v) we also have that $h\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}\right) < 0$, which proves existence and uniqueness of σ_{H_3}. We then have $\epsilon_{\gamma H} > \epsilon_{\gamma F}$ for $\sigma > \sigma_{H_3}$, and $\epsilon_{\gamma H} < \epsilon_{\gamma F}$ for $\sigma < \sigma_{H_3}$. In configuration (vi), on the contrary,

[77] Indeed, this coefficient is given by

$$\begin{aligned} c \;\equiv\; & (2+\theta\alpha_{KK})\{\alpha_{LL} - \alpha_{\Gamma L} + \theta[\alpha_{\Gamma K}\alpha_{KL} - \alpha_{KK}(1+\alpha_{\Gamma L})]\} \\ & +2(1+\theta\alpha_{KK})\{2+\alpha_{LL}+\alpha_{\Gamma L} - \theta[\alpha_{\Gamma K}\alpha_{KL} - \alpha_{KK}(1+\alpha_{LL})]\}\end{aligned}$$

In configuration (iv) $c > 0$, by Assumption 1 and 2. In configurations (v) and (vi) it is also positive since $\alpha_{Ki} = \beta_{Ki} = 0$ and c becomes $c \equiv 4(1+\alpha_{LL})$ which is positive by Assumption 1.

$h\left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}\right) > 0$. Therefore two cases are possible. Either there are two roots, $\sigma^a_{H_3}$ and $\sigma^b_{H_3}$, for the polynomial $h(\sigma_{H_3}) = 0$, such that $\frac{s-\beta_{LL}}{1+\alpha_{LL}} < \sigma^a_{H_3} < \sigma^b_{H_3} < \sigma_F$, and in this case $\varepsilon_{\gamma_H} < \varepsilon_{\gamma_F}$ for $\sigma \in \left(\sigma_{H_3^a}, \sigma_{H_3^b}\right)$, and $\varepsilon_{\gamma_H} > \varepsilon_{\gamma_F}$ otherwise. Notice however that the existence of $\sigma_{H_3} \in \left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_F\right)$ in this configuration requires that $S > S_1$, which is ruled out by Assumption 7. Alternatively there is no $\sigma_{H_3} \in \left(\frac{s-\beta_{LL}}{1+\alpha_{LL}}, \sigma_F\right)$ and $\varepsilon_{\gamma H} > \varepsilon_{\gamma_F}$ for all $\sigma > \frac{s-\beta_{LL}}{1+\alpha_{LL}}$.

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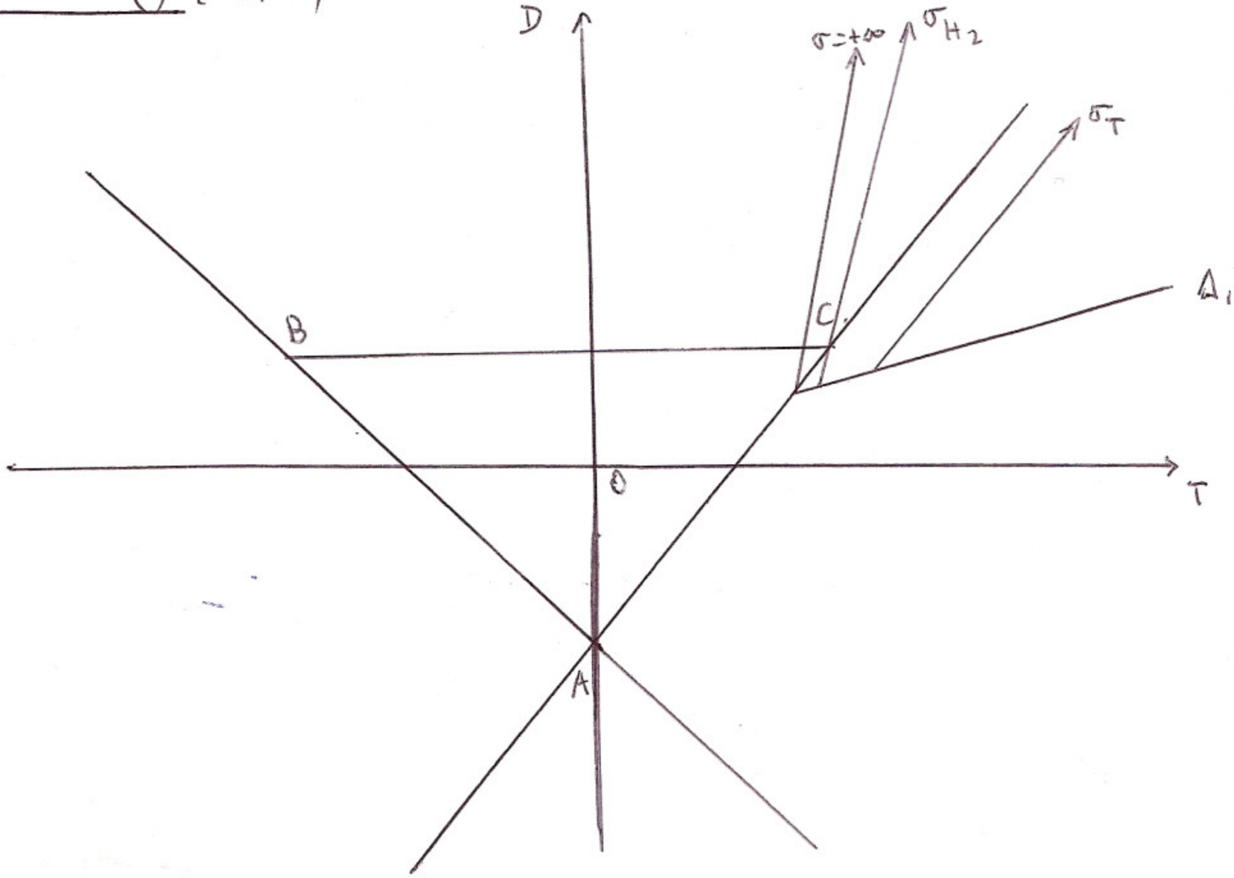
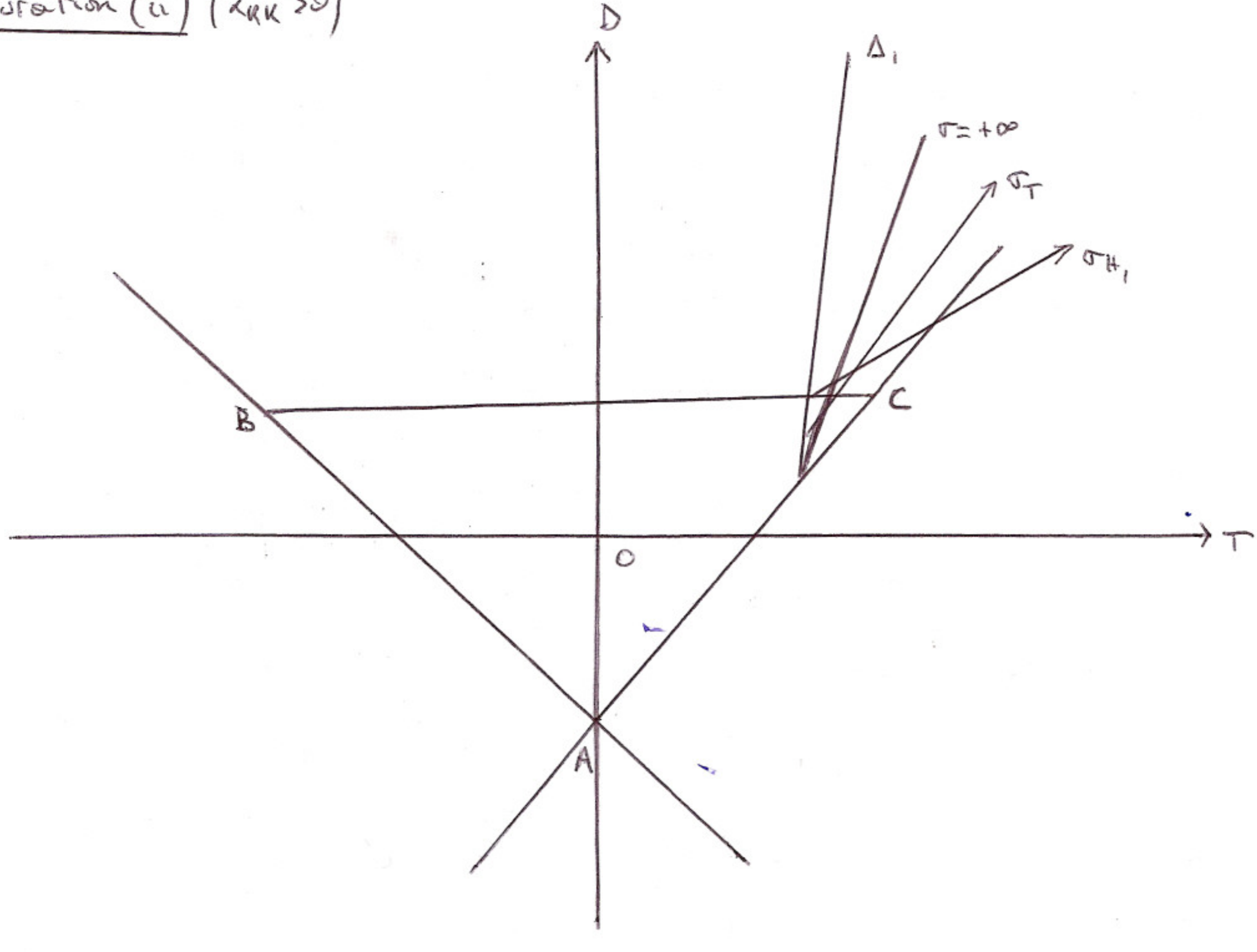
Configuration (i) ($\alpha_{KK} > 0$)



Configuration (ii) ($\alpha_{KK} > 0$)



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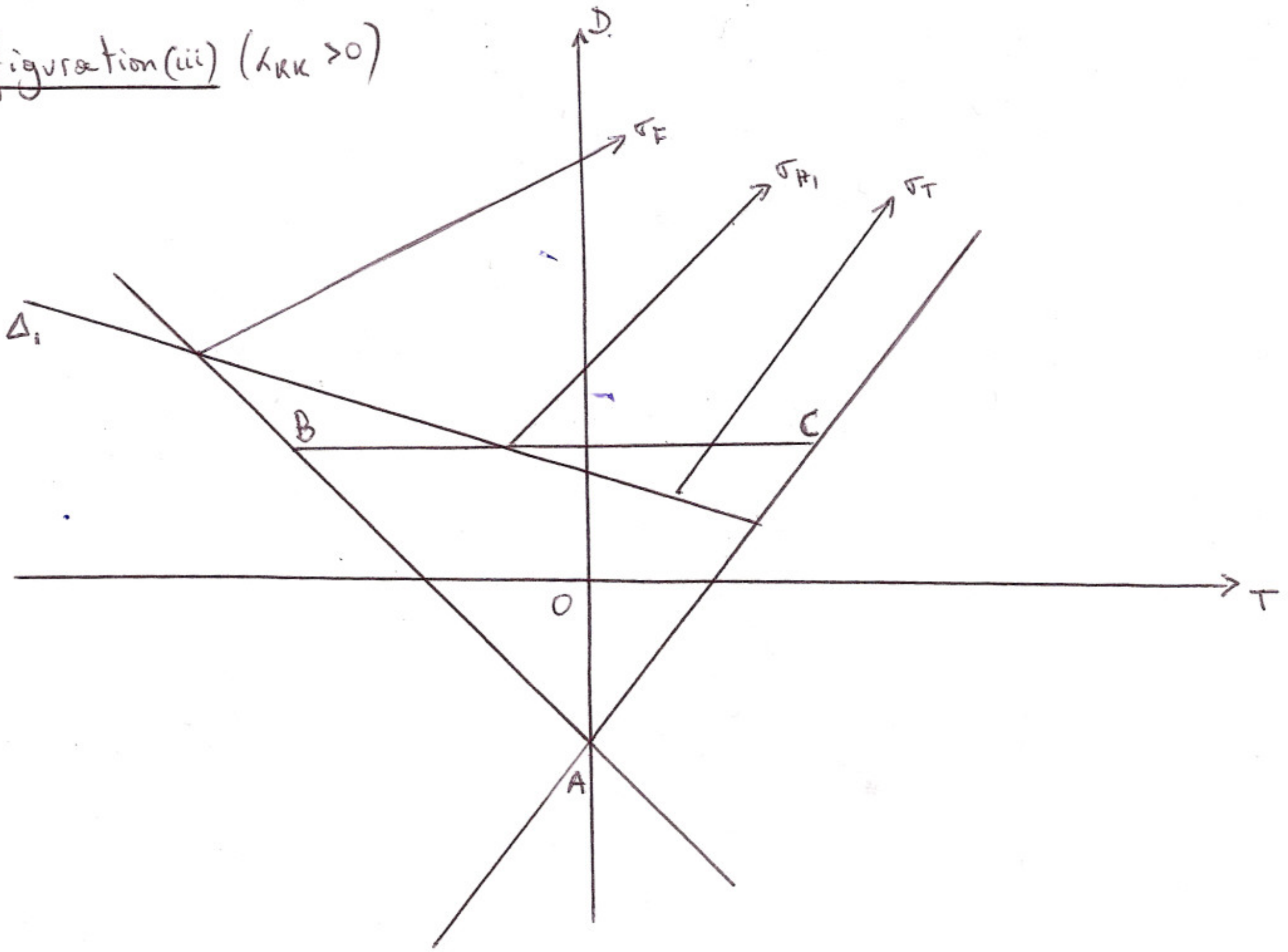
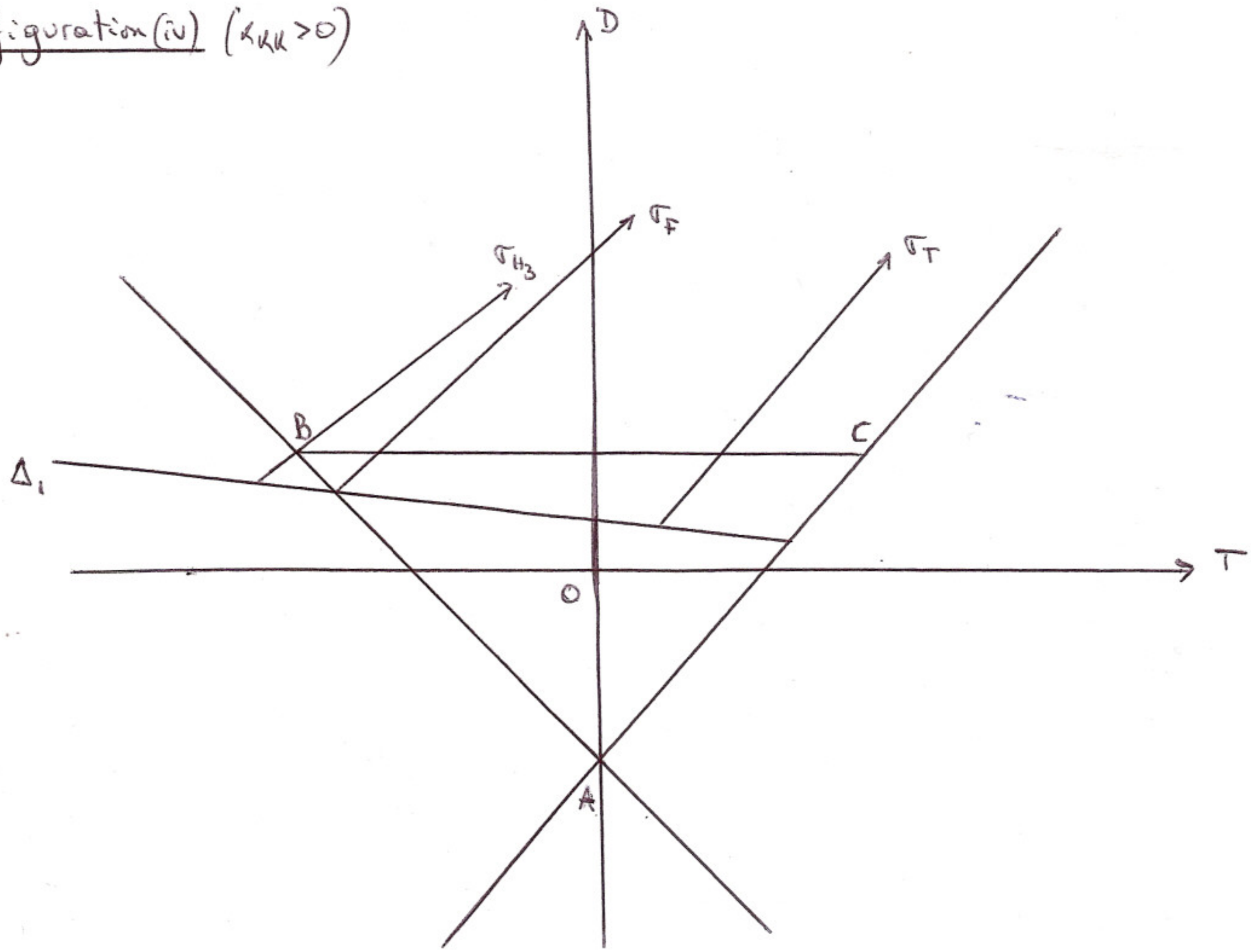


Configuration (iv) ($\lambda_{KK} > 0$)



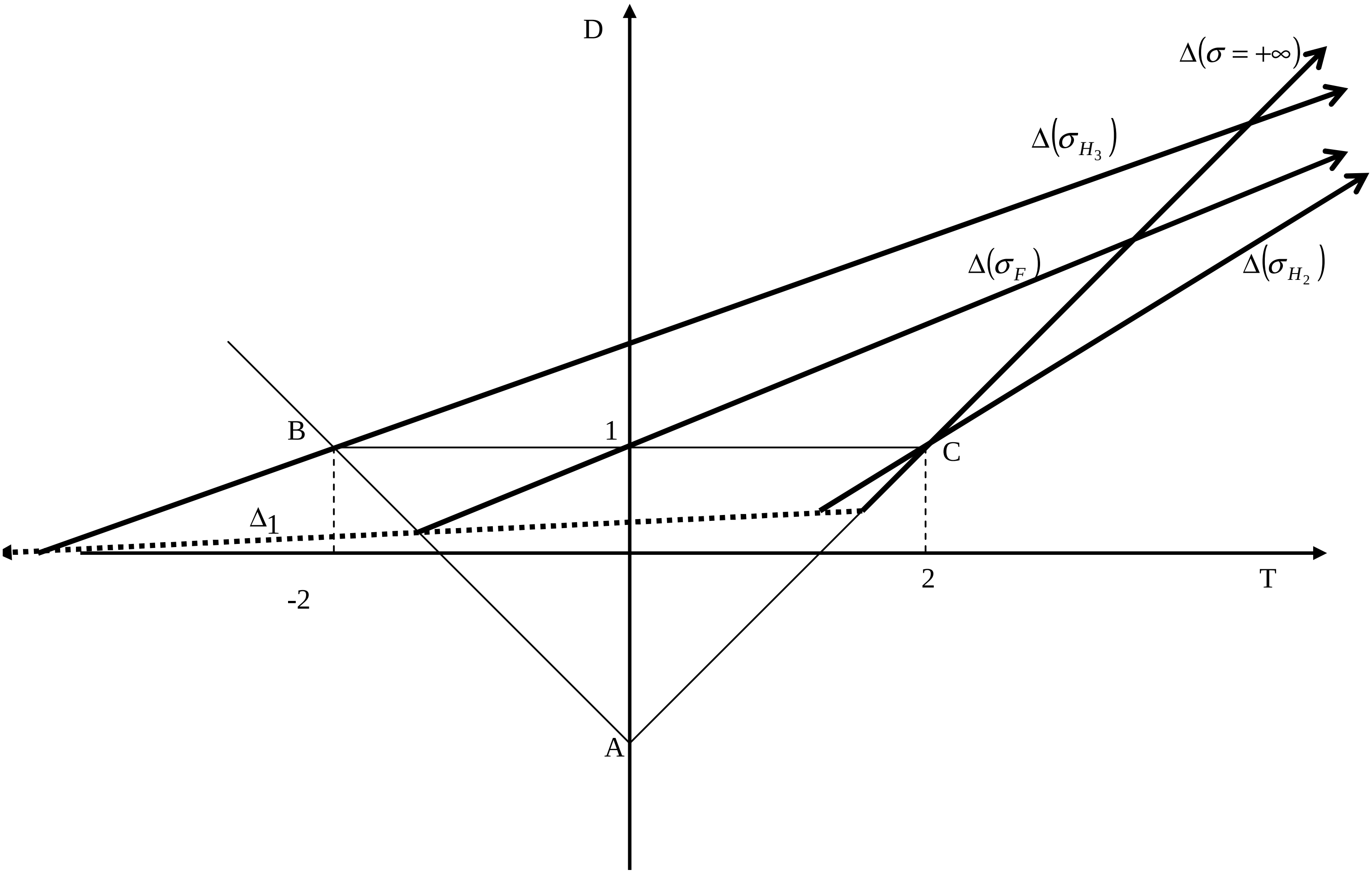


Figure 5: Configuration (v), Proposition 6.1

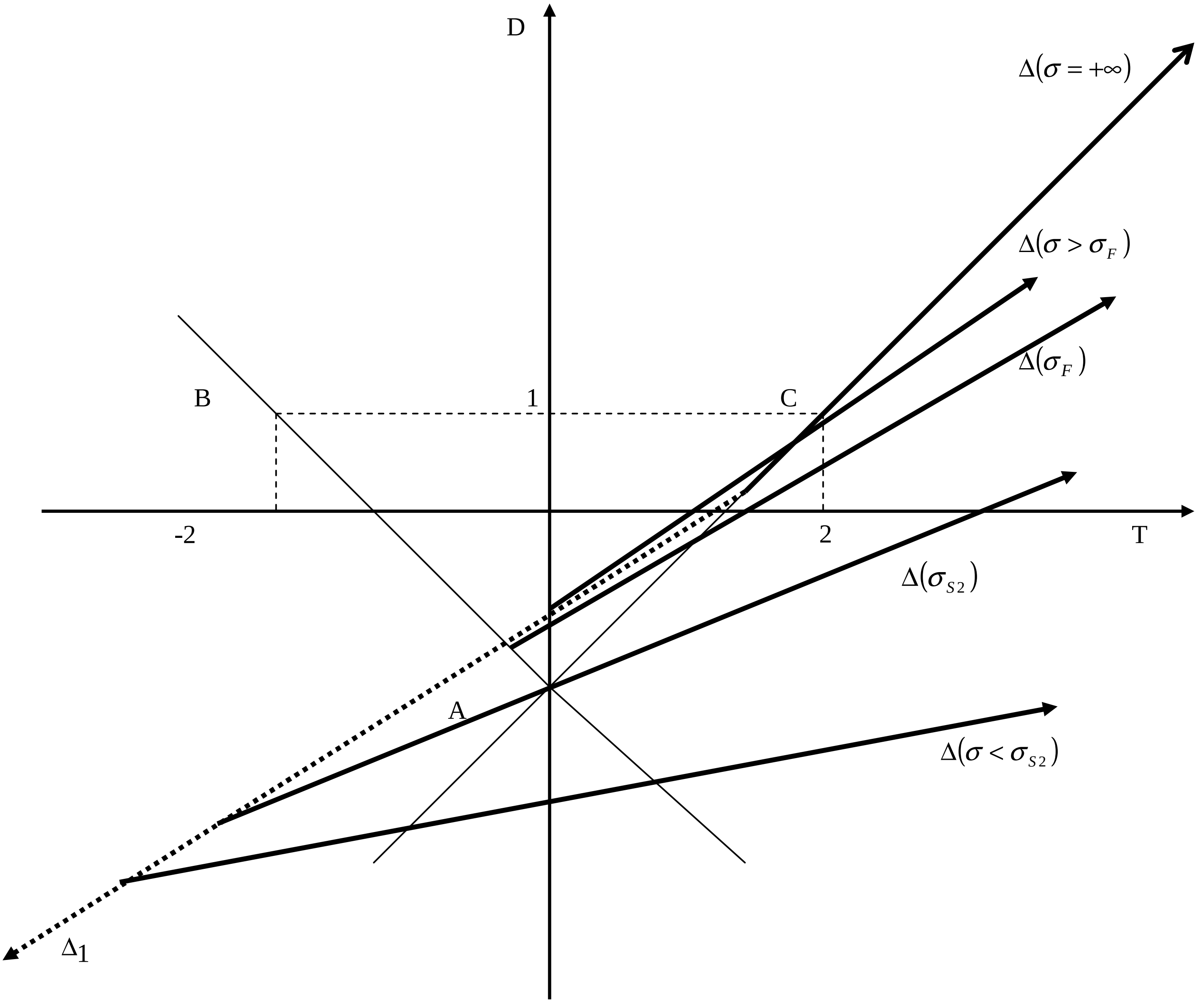


Figure 6: Configuration (*vi*), Proposition 7.1