Money, liquidity and the equilibrium interest rate*

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Abstract

This paper characterizes a random matching model where fiat money and risk-free nominal bonds coexist as competing media of exchange. We introduce explicit frictions in the Lagos and Wright (2005)'s model, frictions that allow us to endogenize bonds acceptability. We derive equilibrium conditions such that the nominal interest rate varies within the interval $[0, u'(q)/c'(q)-1]$ depending on the relative liquidity of these assets.

Keywords: Money, bonds, liquidity, counterfeiting, nominal interest rates

JEL Classification: E40, H20, H63

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1 Introduction

Some categories of nominal government-issued securities share the features that characterize money. Namely, they are payable-to-the-bearer and virtually default-free.[1] Money is the generally accepted medium of exchange in most economies, however, despite the fact that nominal bonds yield their bearers a greater rate of return. This seems to violate a simple no-arbitrage condition: Why is it that money coexists with bonds exhibiting dominance in the rate of return?

The violation of this no-arbitrage condition, which is commonly referred to as the 'rate-of-return dominance of money' (or 'coexistence puzzle'), has been a challenging issue in monetary economics and dates back at least as far as [13]. It is often argued that riskless nominal bonds rate-of-return dominate money because they are less liquid, namely the nominal interest rate on securities is a compensation for the relative illiquidity of these assets.[2] This argument gives rise the following question: Why are bonds less liquid than money? To put it different, What kind of frictions make nominal bonds less liquid than money?

An answer to this question can be found in [16]: nominal bonds are not, because of physical or informational reasons, as attractive as cash in goods trade. For example, today the minimum size of U.S. Treasury-bills (T-bills) is $1,000 which is greater than the value of many transactions.[3] Large denomination is not sufficient, however, to preclude bonds from circulating as a medium of exchange. Financial intermediaries can convert large-denominated government-issued securities into small-denominated privately-issued securities —securities that are safe by virtue of being backed by the underlying securities. Thus, in absence of the cost of such intermediation, the coexistence puzzle may be explained by legal restrictions that prevent arbitrage between T-bills and currency-like assets. (A detailed discussion on this can be found in [1], [3], [9], and [27].) Another way to have money and bonds coexist is to restrict bonds acceptability.

This paper studies the coexistence of money and bonds by using a framework in which the role of money is formalized explicitly. When the buyers' cash constraint is binding, the equilibrium nominal interest rate is an average between zero and the liquidity premium on money with weights distributed according to the degree of liquidity of bonds. When the cash constraint is not binding

[1] One notable example is the issuance of 'bons' by the French government during the period 1915-1927 (see [23]).
[2] See, for example, [1], [3], [14], [17], [26], and [29]. For a more general discussion see [18].
[3] Moreover, T-bills are entirely book-entry securities and no tangible certificate is given to investors.

the interest rate is zero, and bonds and money are essentially the same thing (means of payments.) We show that money and bonds are competing media of exchange when they have the same value at the margin, and the equilibrium interest rate depends on the cost of being informed.

Initially, we assume that bonds acceptability is exogenous. We then endogenize acceptability by introducing additional frictions into the baseline model. We derive equilibrium conditions such that the nominal interest rate varies within a given interval depending on the intensity of these frictions.

A number of papers study liquidity issues. [19] examine an economy where money and real assets are competing media of exchange. They show that, in the absence of money, the economy over-accumulates capital if the stock real assets is too low to sustain efficient trade. In this case, money is welfare improving in that it reduces capital overaccumulation. [17] builds an asset-pricing model in which financial assets (equity shares and government risk-free real bonds) are valued for their liquidity, not only for the stream of consumption goods that they represent. They show that the price of an asset will be higher when the asset is held for its liquidity value, and its rate of return will be lower than it would be if the asset was not used for payment. In a different but related paper, [10] investigate the effect of monetary policy on prices and allocations. They show that money is valued if and only if real assets are scarce, in which case money and real assets compete as means of payment. When inflation increases (i.e. the return on money decreases), a no arbitrage condition implies that the price of the asset increases in order to lower its real return. [22] extend [10] by modeling different acceptabilities. They endogenize acceptability and show that, in equilibrium, assets have different liquidity properties. Our model allows for competing media of exchange but, unlike previous works, extends the analysis in two directions. First, we focus on the competition between money and nominal assets (i.e. government risk-free nominal bonds), which allows us to endogenize the nominal interest rate, second, and more importantly, we endogenize the acceptability of these assets by explicitly introducing additional frictions into the baseline model.

In a recent paper, [7] study an economy in which agents hold outside bonds and inside bonds. They show that the optimal allocation with outside bonds dominates the optimal allocation with inside bonds. In their model bonds are illiquid. Here we allow outside bonds to be liquid. In another paper, [16] constructs a model in which agents can trade money for bonds before entering the goods market and after having observed the taste shock. He finds that illiquid bonds are welfare improving, while liquid bonds are inessential.[4] In contrast, we derive differential acceptability and show that (partially) liquid bonds are essential.

The paper is organized as follows. Section 2 describes the basic framework of the model. Section 3 characterizes stationary equilibria and derives the main results of the analysis. Section 4 endogenizes partial acceptability. The Conclusions end the paper.

2 The model

The basic setup is [20] and [24]. Time is indexed by $t \in \mathbb{N}$. In each period t there are three markets that open sequentially. There is a $\mathbb{R}[0,1]$ continuum of infinitely-lived agents, and two types of perishable commodities: general and special goods. Specialization is described as follows. In the second market, an agent meets someone who produces a good he wishes to consume with probability $\sigma \in (0, 1/2]$, and meets someone who likes the good he produces with the same probability σ; with probability $1 - 2\sigma$ he meets no one. This leads to a double-coincidence-of-wants problem: an agent can either produce or consume in a meeting but not both. We refer to consumers as *buyers* and producers as *sellers*; those who neither produce nor consume are *nontraders*.

Buyers get utility $u(q)$ from q consumption of the special good, where $u'(q) > 0$, $u''(q) < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. Producers incur utility cost $c(q)$ from producing q units of output. Let q^* denote the solution to $u'(q^*) = c'(q^*)$.

In the third market all agents consume and produce the general good, getting utility $U(x)$ from x consumption, with $U'(x) > 0$, $U'(0) = \infty$, $U'(\infty) = 0$ and $U''(x) \leq 0$. Let x^* be the solution to $U'(x^*) = 1$. All agents can produce the general good from labor using a linear technology. They discount between market 3 and the next-period market 1, but not between market 1 and market 2 nor between market 2 and market 3. This is not restrictive since all that matters is the total discounting between one period and the next (e.g., [25]).

Individual actions are not observable in the third market so as to avoid contagion equilibria ([2] and [21]). Also, it is assumed that all agents are *anonymous*. Consequently, trade credit is ruled out and transactions are subject to a *quid pro quo* restriction, so there is a role for a medium of exchange ([15] and [28]).

[4] As in [16], other papers like [8] and [26] assume that agents can trade between money and bonds after having observed the preference shocks.

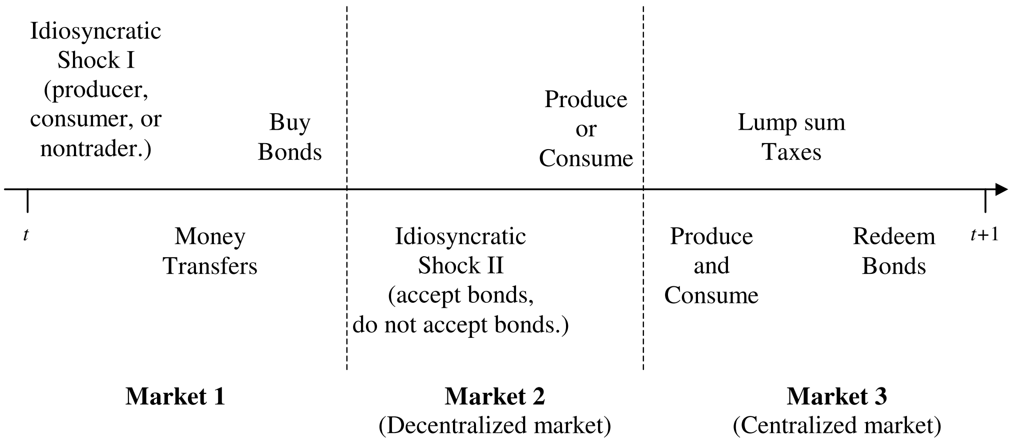

Figure 1: Timing of events

At the opening of the first market, agents learn their type (buyers, sellers or nontraders). After types are revealed, agents can invest their money in a risk-free asset a bearing the gross nominal rate of return $1+i$ with $i \geq 0$. The interest rate i is kept exogenous until Section 4.

Following [29], nominal bonds are one-period risk-free assets that automatically turn into money in market 3. Suppose there are vending machines maintained by the government which offer such bonds in exchange for cash. We assume $a \in \mathbb{R}_+$, so that agents can invest but not borrow. Interest rates on bonds are financed by lump-sum taxes levied by the government in market 3.

There is a central bank that controls the money supply at time t, $M_t > 0$. Also, it is assumed that $M_t = \gamma M_{t-1}$, where $\gamma > 0$ is constant and new money is injected, or withdrawn if $\gamma < 1$, as lump-sum transfers $\pi M_{t-1} = (\gamma - 1) M_{t-1}$ to all agents. We restrict attention to policies where $\gamma \geq \beta$, with $\beta \in \mathbb{R}\,(0,1)$ denoting the discount factor. The time subscript t is omitted and shorten $t+1$ to $+1$, etc. in what follows.

The timing of events is shown in Figure 1. At the beginning of market 1 agents observe their type and receive the lump-sum money transfers. Then, vending machines are activated and individuals

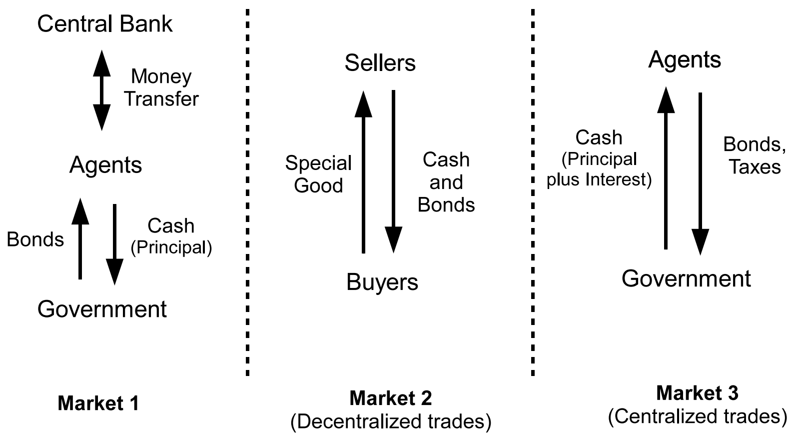

Figure 2: Environment

have the opportunity to invest their cash in nominal bonds. After investment decisions have been made, bond vending machines get disabled and the second market opens. In the second market, agents learn whether or not they will be able to accept nominal bonds for payment in bilateral meetings. In the third market, they produce and consume the general good, pay taxes, and receive the principal plus interest on bonds. The structure of this economy is shown in Figure 2.

Let $\phi = 1/P$ be the real price of money and P the price of goods in market 3. We study steady state equilibria, where aggregate real money balances are constant. We refer to this as stationary equilibrium

$$\phi M = \phi_{-1} M_{-1} \tag{1}$$

which implies $\phi_{-1}/\phi = M/M_{-1} = \gamma$.

In nominal terms, we define the government budget constraint as follows

$$PG + Ai = T \tag{2}$$

where A is the risk-free asset (government debt) outstanding at the opening of market 3, T is a lump-sum nominal tax, and PG is spending for government consumption. Equation (2) states that the government expenditure ($PG + Ai$) is financed by tax revenues (T). To simplify the analysis, we assume $G = 0$.

3 Value functions

In what follows, we look at a representative period t and work backwards from the third to the first market.

Consider a stationary equilibrium. In the third market agents produce h units of the general good using h hours of labor, pay taxes, receive repayment of the investment plus interest, consume x units of the general good, and adjust their money balances. The real wage per hour is normalized to one. Let $V_1(m_1)$ denote the expected value from entering market 1 with m_1 money balances, and $V_3(m_3, a_3)$ the expected value from entering the third market with m_3 units of money and a_3 bonds. Hence, the representative agent's problem in market 3 is

$$V_3\left(m_3, a_3\right) = \max_{x,h,m_{1,+1}} \left[U\left(x\right) - h + \beta V_{1,+1}\left(m_{1,+1}\right)\right] \tag{3}$$

such that

$$x = h + \phi\left(m_3 - m_{1,+1}\right) + \phi\left(1+i\right)a_3 - \phi T \tag{4}$$

where $m_{1,+1}$ is the money taken into period $t+1$. Substituting h from (3) into (4) yields

$$\begin{aligned} V_3\left(m_3, a_3\right) = {} & \phi\left[m_3 + \left(1+i\right)a_3 - T\right] \\ & + \max_{x,m_{1,+1}} \left[U\left(x\right) - x - \phi m_{1,+1} + \beta V_{1,+1}\left(m_{1,+1}\right)\right]. \end{aligned} \tag{5}$$

The first order conditions (FOCs) for (5) are

$$\begin{aligned} U'\left(x\right) &= 1, \\ \beta V'_{1,+1}\left(m_{1,+1}\right) &= \phi, \end{aligned} \tag{6}$$

where $\beta V'_{1,+1}\left(m_{1,+1}\right)$ is the marginal benefit of taking money into the next period, and ϕ is its marginal cost. Due to concavity of u, $m_{1,+1}$ is unique; so all agents exit the third market with the same amount of cash.

Two results from (6) are familiar with [20]: (i) agents consume the efficient quantity x^* of the general good in market 3, where x^* satisfies $U'\left(x^*\right) = 1$; and (ii) $m_{1,+1}$ is independent of a_3 and m_3; thus, the distribution of money holdings is degenerate at the beginning of the next period. This is due to the quasi-linearity hypothesis in (3).

From the envelope theorem, it holds that

$$\begin{aligned} V_3^m &= \phi, \\ V_3^a &= \phi\,(1+i)\,. \end{aligned} \tag{7}$$

In market 2, agents are subject to an idiosyncratic shock: with probability $\alpha \in [0,1]$ they can use both money and bonds for payment in goods trade, while with probability $1-\alpha$ they can use only money. Hence, α is a measure of the degree of liquidity of bonds. Let us denote 'type-II meeting' a meeting in which both assets (money and bonds) can be used for payment, and 'type-I meeting' a meeting in which only money is accepted. The measure of the liquidity of bonds, α, is kept exogenous until Section 4.

Let $V_{2,b}(m_{2,b}, a_{2,b})$ be the value function of a buyer who enters the second market with $m_{2,b}$ units of money and $a_{2,b}$ bonds; $V_{2,\ell}(m_{2,\ell}, a_{2,\ell})$ is the value function for an agent who is not a buyer (i.e. he is either a seller or nontrader), and has $m_{2,\ell}$ units of money and $a_{2,\ell}$ bonds at the beginning of market 2. We refer to sellers and nontraders as *nonbuyers.* Hence, the expected utility for an agent entering the second market is:

$$V_2(m_{2,b}, a_{2,b}, m_{2,\ell}, a_{2,\ell}) = \sigma V_{2,b}(m_{2,b}, a_{2,b}) + (1-\sigma)\, V_{2,\ell}(m_{2,\ell}, a_{2,\ell}) \tag{8}$$

where

$$\begin{aligned} V_{2,b}(m_{2,b}, a_{2,b}) = {} & \alpha \left[u\left(q_{II}\right) + V_3\left(m_{2,b} - z_{m,b}, a_{2,b} - z_{a,b}\right)\right] \\ & + (1-\alpha)\left[u\left(q_I\right) + V_3\left(m_{2,b} - z_{m,b}, a_{2,b}\right)\right] \end{aligned} \tag{9}$$

and

$$\begin{aligned} V_{2,\ell}(m_{2,\ell}, a_{2,\ell}) = {} & \tfrac{\sigma\alpha}{1-\sigma}\left[-c\left(q_{II}\right) + V_3\left(m_{2,\ell} + z_{m,s}, a_{2,\ell} + z_{a,s}\right)\right] \\ & + \tfrac{\sigma(1-\alpha)}{1-\sigma}\left[-c\left(q_I\right) + V_3\left(m_{2,\ell} + z_{m,s}, a_{2,\ell}\right)\right] \\ & + \tfrac{1-2\sigma}{1-\sigma} V_3\left(m_{2,\ell}, a_{2,\ell}\right). \end{aligned} \tag{10}$$

Expression (9) indicates that: (i) with probability α a buyer is in a type-II meeting, which means that he consumes a quantity q_{II} of goods, pays $z_{m,b}$ amount of money, and gives up $z_{a,b}$ bonds (first line); (ii) with probability $1-\alpha$ he is in a type-I meeting, which means that he consumes q_I units of goods, and spends $z_{m,b}$ amount of money (second line). The value function (10) means that three events may occur for a nonbuyer in market 2: (i) with probability $\sigma\alpha/\left(1-\sigma\right)$ she is a producer in a type-II meeting, in which case she produces a quantity q_{II} of the special good, receives $z_{m,s}$ amount of money and $z_{a,s}$ bonds (first line); (ii) with probability $\sigma(1-\alpha)/(1-\sigma)$ she is a producer in a type-I meeting, which means that she produces q_I units of the special good, and receives $z_{m,s}$ amount of money (second line); (iii) with probability $(1-2\sigma)/(1-\sigma)$ she is a nontrader (third line).

We assume that the terms of trade $\{q_I, q_{II}, z_m, z_a\}$ are determined by generalized Nash bargaining.[5] Let θ denote the bargaining power of the seller, with $z_m = z_{m,b} = z_{m,s}$ and $z_a = z_{a,b} = z_{a,s}$. Then, $\{q_{II}, z_m, z_a\}$ solves

$$\begin{aligned} &\max_{q_{II}, z_m, z_a} \left[-c\left(q_{II}\right) + \phi z_m + \phi\left(1+i\right) z_a\right]^{\theta}\left[u\left(q_{II}\right) - \phi z_m - \phi\left(1+i\right) z_a\right]^{1-\theta} \\ &\qquad s.t. \\ &\qquad\qquad z_m \leq m_{2,b} \\ &\qquad\qquad z_a \leq a_{2,b}, \end{aligned} \tag{11}$$

when in a type-II meeting, and $\{q_I, z_m\}$ solves

$$\begin{aligned} &\max_{q_I, z_m} \left[-c\left(q_I\right) + \phi z_m\right]^{\theta}\left[u\left(q_I\right) - \phi z_m\right]^{1-\theta} \\ &\qquad s.t. \\ &\qquad\qquad z_m \leq m_{2,b}, \end{aligned} \tag{12}$$

when in a type-I meeting.

From (11), the net gains from trade for buyers and sellers in type-II meetings are the following. For a seller, the surplus from trade is $-c\left(q_{II}\right) + V_3(m_{2,\ell} + z_m, a_{2,\ell} + z_a)$, and threat point given by her continuation value $V_3(m_{2,\ell}, a_{2,\ell})$. So, using linearity of V_3, her net surplus is $-c\left(q_{II}\right) + \phi z_m + \phi\left(1+i\right) z_a$. For a buyer, the gain from trade is $u\left(q_{II}\right) + V_3(m_{2,b} - z_m, a_{2,b} - z_a)$, and the threat point $V_3(m_{2,b}, a_{2,b})$. Again, by linearity of V_3, his net surplus is $u\left(q_{II}\right) - \phi z_m - \phi\left(1+i\right) z_a$. The first constraint in (11) means that a buyer in a type-II meeting cannot spend more cash than what he brings into market 2. The second constraint means that he cannot give up more bonds than what he has in his portfolio. In type-I meetings bonds are not accepted for payment, so the net surplus from trade is $-c\left(q_I\right) + \phi z_m$ for sellers, and $u\left(q_I\right) - \phi z_m$ for buyers. As usual in [20]-like models,

[5] See [25] for alternative price mechanisms (price posting, competitive search).

the terms of trade, denoted as $\{q_{II}, z_m, z_a\}$ and $\{q_I, z_m\}$, depend on the buyer's portfolio of assets, $\{m_{2,b}, a_{2,b}\}$, and they do not depend on the seller's, $\{m_{2,s}, a_{2,s}\}$.

The solution to (11) and (12) are

$$\phi z_m + \phi (1+i) z_a = g(q_{II}) \equiv \frac{\theta u'(q_{II}) c(q_{II}) + (1-\theta) u(q_{II}) c'(q_{II})}{\theta u(q_{II}) + (1-\theta) c'(q_{II})} \tag{13}$$

and

$$\phi z_m = g(q_I) \equiv \frac{\theta u'(q_I) c(q_I) + (1-\theta) u(q_I) c'(q_I)}{\theta u(q_I) + (1-\theta) c'(q_I)}, \tag{14}$$

respectively. Note that constraints in (11) and (12) always bind; it is easy to show that buyers spend all their money in type-II and type-I meetings, and give up all their bonds in type-II meetings. Thus, assuming that the sellers have all the bargaining power ($\theta = 1$), expressions (13) and (14) can be rewritten as follows:

$$\phi m_{2,b} + \phi (1+i) a_{2,b} = g(q_{II}) = c(q_{II}) \tag{15}$$

and

$$\phi m_{2,b} = g(q_I) = c(q_I). \tag{16}$$

Before analyzing the agent's problem in market 1 we have to derive the marginal value of money and the marginal value of bonds, for buyers and nonbuyers, in market 2. Let us consider buyers first. Take the differential of $V_{2,b}(m_{2,b}, a_{2,b})$ with respect to $m_{2,b}$ and get

$$V_{2,b}^m = \phi \left\{ \alpha \frac{u'(q_{II})}{c'(q_{II})} + (1-\alpha) \frac{u'(q_I)}{c'(q_I)} \right\} \tag{17}$$

where we have used $\partial q_{II}/\partial m_{2,b} = \phi/c'(q_{II})$ from (15), $\partial q_I/\partial m_{2,b} = \phi/c'(q_I)$ from (16), and the fact that buyers in market 2 spend all their money holdings, i.e. $\partial z_{m,b}/\partial m_{2,b} = 1$. Expression (17) means that the marginal value of money for a buyer in market 2 is given by his marginal benefit from consumption when in a type-II meeting (first term), plus his marginal benefit from consumption when in a type-I meeting (second term). So, the full return on money for a buyer in market 2 equals its liquidity return, return that depends on the extent to which money is accepted as a medium of exchange.

Now, take the differential of $V_{2,b}(m_{2,b}, a_{2,b})$ with respect to $a_{2,b}$ and get

$$V_{2,b}^{a} = \phi\left(1+i\right)\left\{\alpha\frac{u'\left(q_{II}\right)}{c'\left(q_{II}\right)} + \left(1-\alpha\right)\right\} \tag{18}$$

where use of $\partial q_{II}/\partial a_{2,b} = \phi\left(1+i\right)/c'\left(q_{II}\right)$ from (15) has been made. Since buyers in a type-II meeting spend all their bond holdings, then $\partial z_{a,b}/\partial a_{2,b} = 1$. Expression (18) indicates that the full return on bonds for a buyer in market 2 is given by two components: (i) the liquidity return, i.e. the asset's usefulness to facilitate trade in type-II meetings, and (ii) the intrinsic rate of return, i.e. the nominal interest rate i they yield at the end of each period.

For an agent who is not a buyer in the second market, the marginal value of money and the marginal value of bonds must be equal to the marginal benefit of carrying these objects into the third market:

$$V_{2,\ell}^{m} = \phi \tag{19}$$

and

$$V_{2,\ell}^{a} = \phi\left(1+i\right). \tag{20}$$

Since nonbuyers cannot consume special goods they do not receive any consumption gain from having more units of money or more bonds in market 2.

At the opening of market 1, new money is injected (or withdrawn) by lump sum transfers. Then, agents realize whether they will be buyers or nonbuyers in the next market 2. After types are revealed, bond vending machines are activated and investment decisions can be made. The agent's expected utility of entering market 1 with m_1 units of money is

$$\begin{aligned} V_1(m_1) = {} & \sigma V_{2,b}(m_1 + \pi M_{-1} - a_{1,b}, a_{1,b}) \\ & + \left(1-\sigma\right) V_{2,\ell}(m_1 + \pi M_{-1} - a_{1,\ell}, a_{1,\ell}), \end{aligned} \tag{21}$$

which means that he is a buyer, with probability σ, in which case he will invest $a_{1,b}$ amount of money buying $a_{1,b}$ bonds (first line); he is a nonbuyer, with probability $1-\sigma$, which means that he will invest $a_{1,\ell}$ amount of money buying $a_{1,\ell}$ bonds (second line). We now derive the optimal choice of $a_{1,b}$ and $a_{1,\ell}$ for buyers and nonbuyers, respectively. (Note that the quantity of bonds acquired in market 1 is carried forward into market 2, so $a_{1,b} = a_{2,b}$ and $a_{1,\ell} = a_{2,\ell}$.)

The buyer's problem in market 1 is as follows:

$$
\begin{aligned}
&\max_{a_{1,b}} V_{2,b}(m_1 + \pi M_{-1} - a_{1,b}, a_{1,b}) \\
&s.t. \\
&\quad m_1 + \pi M_{-1} - a_{1,b} \geq 0 \\
&\quad a_{1,b} \geq 0,
\end{aligned}
\tag{22}
$$

where the first constraint means that he cannot invest more cash than what he brings into the first market, m_1, plus the transfer πM_{-1}. The second constraint means that he cannot sell bonds (i.e. he cannot borrow). The FOCs for (22) are:

$$
\begin{aligned}
V_{2,b}^a - V_{2,b}^m - \mu_b + \zeta_b = 0 \\
\mu_b \left(m_1 + \pi M_{-1} - a_{1,b} \right) = 0 \\
\zeta_b a_{1,b} = 0,
\end{aligned}
\tag{23}
$$

where μ_b and ζ_b are the Lagrange multipliers for the first, respectively the second, constraint in (22). Of course, both constraints in (22) cannot bind simultaneously, which means that at least one multiplier must be zero.

Lemma 1 *If $V_{2,b}^m = V_{2,b}^a$, then $a_{1,b} \in [0, m_1 + \pi M_{-1}]$.*

Proof. Assume $V_{2,b}^a = V_{2,b}^m$. Then, it holds that $\mu_b = 0$ and $\zeta_b = 0$, so $a_{1,b} \in [0, m_1 + \pi M_{-1}]$. ■

Buyers make their investment decisions in market 1 on the basis of the next-market-2's marginal values of money and bonds, $V_{2,b}^m$ and $V_{2,b}^a$. If $V_{2,b}^a = V_{2,b}^m$, buyers in market 1 are indifferent (they randomize) between buying bonds and holding cash, so $a_{1,b} \in [0, m_1 + \pi M_{-1}]$.

The following result can now be established:

Proposition 1 *If $V_{2,b}^m = V_{2,b}^a$, money and bonds are competing media of exchange in market 2.*

Proof. Assume $V_{2,b}^m = V_{2,b}^a$. Then, a buyer in market 1 is indifferent between investing and not investing, so he will enter market 2 with a positive amount of cash and bonds. ■

The problem for a nonbuyer in the first market is:

$$\begin{aligned}
&\max_{a_{1,\ell}} V_{2,\ell}(m_1 + \pi M_{-1} - a_{1,\ell}, a_{1,\ell}) \\
&s.t. \\
&\qquad m_1 + \pi M_{-1} - a_{1,\ell} \geq 0 \\
&\qquad a_{1,\ell} \geq 0,
\end{aligned} \tag{24}$$

with FOCs

$$\begin{aligned}
V_{2,\ell}^a - V_{2,\ell}^m - \mu_\ell + \zeta_\ell = 0 \\
\mu_\ell \left(m_1 + \pi M_{-1} - a_{1,\ell}\right) = 0 \\
\zeta_\ell a_{1,\ell} = 0,
\end{aligned} \tag{25}$$

where μ_ℓ and ζ_ℓ are the Lagrange multipliers for the first, respectively the second, constraint in (24). Again, at least one multiplier must be zero. If the nominal interest is strictly positive, nonbuyers will always invest all their money in nominal bonds, i.e. the first constraint in (24) binds. To see this note that $V_{2,\ell}^a > V_{2,\ell}^m$ iff $i > 0$, directly from (19), (20) and (25). This observation is very intuitive if not obvious: money and bonds are "illiquid" objects for nonbuyers in the second market, but bonds, rather than money, yield a strictly positive interest rate if $i > 0$.

Let us now derive the marginal value of money for an agent in market 1. Take the differential of (21) with respect to m_1 and get

$$V_1'(m_1) = \sigma \left\{ V_{2,b}^m \left[1 - \frac{\partial a_{1,b}}{\partial m_1} \right] + V_{2,b}^a \frac{\partial a_{1,b}}{\partial m_1} \right\} + (1 - \sigma) V_{2,\ell}^a. \tag{26}$$

Expression (26) means that buyers in market 1 carry a fraction $1 - \partial a_{1,b} \backslash \partial m_1$ of additional unit of money into the second market (first term); they spend the complement fraction $\partial a_{1,b} \backslash \partial m_1$ buying bonds (second term). The fraction of money invested in bonds ($\partial a_{1,b} \backslash \partial m_1$) depends on the buyer's marginal values of money and bonds in market 2, i.e. $V_{2,b}^m$ and $V_{2,b}^a$. By Lemma 1, if $V_{2,b}^m = V_{2,b}^a$, then $\partial a_{1,b} \backslash \partial m_1 \in \mathbb{R}[0, 1]$. The third term in (26) refers to nonbuyers and means that they spend the additional unit of money buying bonds (i.e. $\partial a_{1,\ell} \backslash \partial m_1 = 1$) if $i > 0$. We can now define the equilibrium:

Definition 1 *A monetary equilibrium is a time path for asset prices* $\{\phi, \phi(1 + i)\}$*, asset holdings*

$\{m_1, m_{2,b}, m_{2,\ell}, m_3, a_{1,b}, a_{1,\ell}, a_{2,b}, a_{2,\ell}, a_3\}$, the second market terms of trade $\{q_{II}, q_I, z_m, z_a\}$, and the third market allocation $\{x, h\}$, satisfying (5), (11), (12), (22) and (24), with $\phi > 0$, $\phi(1+i) > 0$, and $m_1 > 0$. In steady state, real variables are constant over time, and (1) holds.

At this point of the analysis, we are ready to endogenize the nominal interest rate. To do this we impose the equilibrium condition such that money and bonds coexist as competing media of exchange, $V_{2,b}^a = V_{2,b}^m$, and solve for i. By (17) and (18), it follows

$$\phi\,(1+i)\left\{\alpha\frac{u'(q_{II})}{c'(q_{II})} + (1-\alpha)\right\} = \phi\left\{\alpha\frac{u'(q_{II})}{c'(q_{II})} + (1-\alpha)\frac{u'(q_I)}{c'(q_I)}\right\}$$

or, rearranging terms and simplifying,

$$i = \frac{\alpha\frac{u'(q_{II})}{c'(q_{II})} + (1-\alpha)\frac{u'(q_I)}{c'(q_I)}}{\alpha\frac{u'(q_{II})}{c'(q_{II})} + (1-\alpha)} - 1. \tag{27}$$

The equilibrium interest rate is the excess liquidity return on money over bonds, excess liquidity return which is proportional to the probability of a type-I meeting —as opposed to a type-II meeting. In other words, the equilibrium interest rate is the liquidity premium on government bonds when money and bonds share the same full return at the margin ($V_{2,b}^m = V_{2,b}^a$). Directly from (27), the next result can be established.

Lemma 2 *The following holds:*

(**1**) *If $\alpha = 1$, then $i \equiv \underline{i} = 0$.*

(**2**) *If $\alpha = 0$, then $i \equiv \bar{\imath} = u'(q_I)/c'(q_I) - 1$.*

(**3**) *If $\alpha \in (0,1)$, then $i \in (\underline{i}, \bar{\imath})$.*

Proof. (**Statements 1 and 2.**) Straightforward from substitution into (27).

(**Statement 3.**) This result is direct consequence of the fact that i is continuous and decreasing in α. ■

4 Endogenous acceptability

So far we have assumed that the agents' decision about using or not nominal bonds for payment in market 2 was exogenous. We will now endogenize it by explicitly introducing more frictions in the baseline setup.

Assume that anyone can costlessly produce counterfeit bonds, bonds that are identical to their authentic counterparts but that perish after they change hands. There is a firm —whose shares are equally held by all agents— that can costlessly produce counterfeit detector machines (CDMs), and whose profits are distributed to all agents as dividends at the end of each period. A CDM is a portable object that perfectly verifies the authenticity of bonds in market 2. For simplicity, we assume that CDMs perish immediately at the end of the period. Anyone in market 2 can buy a CDM at a price $\varepsilon \in \mathbb{R}_+$ payable in the next third market. The agents' decision to buy or not a CDM takes place in market 2, before decentralized trade begins. This decision is public information. The latter assumption assures that agents without a CDM do not accept bonds for payment in bilateral trades.[6]

As before, we look at a representative period t and work backwards from the third to the first market. Let $V_3^{\mathcal{C}}$ be the value function of an agent with a CDM in market 2, and $V_3^{\mathcal{N}}$ the value function of an agent without a CDM in market 2. Hence,

$$\begin{aligned} V_3^{\mathcal{C}}(m_3, a_3) &= \phi \left[m_3 + (1+i)\, a_3 - T + D - \varepsilon\right] \\ &\quad + \max_{x, m_{1,+1}} \left[U(x) - x - \phi m_{1,+1} + \beta V_{1,+1}(m_{1,+1})\right], \end{aligned} \tag{28}$$

and

$$\begin{aligned} V_3^{\mathcal{N}}(m_3, a_3) &= \phi \left[m_3 + (1+i)\, a_3 - T + D\right] \\ &\quad + \max_{x, m_{1,+1}} \left[U(x) - x - \phi m_{1,+1} + \beta V_{1,+1}(m_{1,+1})\right]. \end{aligned} \tag{29}$$

From (28), if an agent buys a CDM in market 2, she pays ε and receives the dividend D in market 3. From (29), an agent without a CDM in market 2 only receives the dividend D in market 3. In all other respects the value functions (28)-(29) are the same as those in Section 3. The FOCs for (28)-(29) are given by (6) and envelope conditions by (7).

In nominal terms, the firm's budget constraint in equilibrium is

$$\sigma\alpha\varepsilon = D, \tag{30}$$

[6] To see this, consider a meeting between a buyer and an seller without a CDM. Since the buyer has perfect knowledge about the fact that his partner does not have a CDM, he will only offer counterfeits bonds for payment. But the seller anticipates this, so she will refuse to take anything other than money.

which means that revenues from sales of CDMs are redistributed to shareholders as dividends.

At the opening of market 2, sellers decide whether they buy a CDM or not. (Note that buyers don't need a CDM since they can only consume in market 2). The seller's net benefit from buying a CDM in market 2 is given by his extra surplus from being in a type-II meeting —as opposed to a type-I meeting— minus the cost of the CDM plus the dividends i.e. $-c\left(q_{II}\right) + V_3\left(m_{2,\ell} + z_{m,s}, a_{2,\ell} + z_{a,s}\right) - \left[-c\left(q_I\right) + V_3\left(m_{2,\ell} + z_{m,s}, a_{2,\ell}\right)\right] - \phi\varepsilon + \phi D$. Let ε^* be the level of ε such that the net benefit of acquiring a CDM is zero, i.e.

$$\varepsilon^* = \frac{-c\left(q_{II}\right) + \phi\left(1+i\right) z_{a,s} + c\left(q_I\right)}{\phi\left(1-\sigma\alpha\right)}$$

where we have used (30) and linearity of V_3. Then, we can state the following

Lemma 3 *In market 2:*

(**1**) *if* $\varepsilon < \varepsilon^*$, *then* $\alpha = 1$,

(**2**) *if* $\varepsilon > \varepsilon^*$, *then* $\alpha = 0$,

(**3**) *if* $\varepsilon = \varepsilon^*$, *then* $\alpha \in (0,1)$.

Proof. (Statement 1.) Assume $\varepsilon < \varepsilon^*$. Then the price of a CDM is smaller than its net benefit, which means that a seller buys a CDM. In a symmetric equilibrium all sellers buy CDMs,so bonds are always accepted in market 2 (i.e. $\alpha = 1$).

(Statement 2.) If $\varepsilon > \varepsilon^*$, then the price of a CDM is greater than its net benefit. This implies that a seller does not buy a CDM in market. In a symmetric equilibrium, nobody acquires a CDM which means that bonds are illiquid in market 2 (i.e. $\alpha = 0$).

(Statement 3.) Assume $\varepsilon = \varepsilon^*$. Then the price of a CDM equals its net benefit, which means that a seller is indifferent between buying and not buying a CDM. Hence, in a symmetric equilibrium each seller randomizes: she buys a CDM with probability $\alpha \in (0,1)$, in which case she accept bonds for payment in market 2, she does not buy with probability $1-\alpha$, in which case she does not accept bonds. ■

The next statement can be established:

Proposition 2 *If* $\varepsilon < \varepsilon^*$, *then the equilibrium interest rate is zero and bonds are inessential.*

Proof. Assume $\varepsilon < \varepsilon^*$. Using Lemma 2 and 3, $\varepsilon < \varepsilon^*$ implies $\alpha = 1$, which implies $i = 0$. ■

If a seller buys a CDM, she will always accept bonds for payment. Thus, in a symmetric equilibrium, bonds and money have the same liquidity (i.e. $\alpha = 1$). By (27), this implies that $i = 0$, which means that nobody has the incentive to invest in nominal bonds. Namely, if bonds are perfectly liquid they are inessential. This is a standard result in search-money literature and can be found in [16].

The next Proposition can now be stated:

Proposition 3 *In an economy where bonds and money are competing media of exchange:*

(**1**) *if* $\varepsilon > \varepsilon^*$, *then bonds are illiquid, and the equilibrium interest rate is* $i = u'(q_I)/c'(q_I) - 1$,

(**2**) *if* $\varepsilon = \varepsilon^*$ *bonds are partially illiquid, and the equilibrium interest rate is within the interval* $\mathbb{R}(\underline{i}, \bar{\imath})$.

Proof. (Statement 1.) Assume (27) holds. Also, assume $\varepsilon > \varepsilon^*$. By Lemma 2 and 3, it follows that $\alpha = 0$ and $i = u'(q_I)/c'(q_I) - 1$.

(Statement 2.) Assume (27) holds, and $\varepsilon = \varepsilon^*$. By Lemma 2 and 3, this implies $\alpha \in (0, 1)$ and $i \in \mathbb{R}(\underline{i}, \bar{\imath})$. ■

The underlying idea of Statement 1 can be summarized as follows. A sellers will not buy a CDM if its price is greater than its net benefit. In a symmetric equilibrium, sellers never accept bonds for payment in market 2 (since they do not buy CDM) if $\varepsilon > \varepsilon^*$. That is, if the price of a CDM is greater than its benefit, then bonds are illiquid and the nominal interest rate is equal to $i = u'(q_I)/c'(q_I) - 1$.

Statement 2 of Proposition 3 has the following meaning. If the price of a CDM equals its net benefit (i.e. $\varepsilon = \varepsilon^*$), then the seller is indifferent between buying and not buying a CDM. In a symmetric equilibrium, she buys the CDM with probability $\alpha \in (0, 1)$, while she does not buy it with probability $1 - \alpha$. Hence, nominal bond are (partially) accepted for payment in market 2, so the nominal interest rate must be strictly positive in order to compensate bondholders for the lower liquidity of bonds (relative to money) if $\varepsilon = \varepsilon^*$. In this case, the equilibrium interest rate is within the interval $(0, u'(q_I)/c'(q_I) - 1)$.

At this point of the analysis we can derive hours of work; details are in the appendix. Hours worked in market 3 are

$$h_b = \alpha \{x^* + \phi m_{1,+1} + \phi T - \phi D\} \\ + (1-\alpha) \{x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,b}\}, \tag{31}$$

for buyers, and

$$h_\ell = \tfrac{\alpha\sigma}{1-\sigma} \{x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) (a_{1,\ell} + z_a) - \phi z_m + \phi \varepsilon\} \\ + \tfrac{(1-\alpha)\sigma}{1-\sigma} \{x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,\ell} - \phi z_m\} \\ + \tfrac{1-2\sigma}{1-\sigma} \{x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,\ell}\}, \tag{32}$$

for nonbuyers. Expression (31) means that: (i) with probability α a buyer enters market 3 with no assets, in which case he has to work $x^* + \phi m_{1,+1} + \phi T - \phi D$ hours to consume x^*, bring $m_{1,+1}$ amount of money into the next period, and pay taxes T minus the dividend D; (ii) with probability $1-\alpha$ he enters market 3 with zero amount of cash and $a_{1,b}$ bonds, so he has to work $x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,b}$ hours; observe that $a_{1,b}$ bonds automatically turn into $(1+i) a_{1,b}$ units of money at the end of market 3, so the buyer who was in a type-I meeting —as opposed to a type-II meeting— in market 2 has to work less in market 3 to bring the same amount of cash into the next-period market 1.

Expression (32) refers to a nonbuyer and means that: (i) with probability $\sigma\alpha/(1-\sigma)$ she is in a type-II meeting, which means that she enters market 3 with z_m amount of cash and $(a_{1,\ell} + z_a)$ bonds. Consequently, she has to work $x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) (a_{1,\ell} + z_a) - \phi z_m + \phi \varepsilon$ hours; (ii) with probability $\sigma (1-\alpha) / (1-\sigma)$ the nonbuyer is in type-I meeting, so she enters market 3 with z_m units of cash and $a_{1,\ell}$ bonds. Hence, she has to work $x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,\ell} - \phi z_m$ hours; (iii) with probability $\sigma (1-\alpha) / (1-\sigma)$ the nonbuyer is a nontrader in market 2, which means that she enters market 3 with zero units of money and $a_{1,\ell}$ bonds. Hence, she has to work $x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,\ell}$ hours. (Nonbuyers invest all their cash in nominal bonds in market 1 if the nominal interest rate is strictly positive, so they enter market 3 with at least $a_{1,\ell}$ units of bonds if $i > 0$.)

Using (31) and (32), aggregate hours of work in market 3, $H = \sigma h_b + (1-\sigma) h_\ell$, are equal to

$$H = x^* + \phi T - \phi i A, \tag{33}$$

by virtue of (30) , $m_{1,+1} = m_1 + \pi M_{-1} = a_{2,b} + m_{2,b} = M$, and using the fact that nonbuyers invest all their money buying bonds in market 1, i.e. $a_{2,\ell} = m_1 + \pi M_{-1} = M$.

Now, eliminate A from (33) using (2), and impose the symmetric condition $x = X$ to get

$$H = X^* \tag{34}$$

with X^* satisfying $U'(X^*) = 1$.

5 Conclusions

This paper studied the coexistence of money and bonds by using a framework in which the role of money is formalized explicitly. We show that money and bonds are competing media of exchange when they have the same value at the margin, and the equilibrium interest rate depends on the price of counterfeit detectors.

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Technical appendix

Derivation of $V^m_{2,b}, V^a_{2,b}, V^m_{2,\ell}$ and $V^a_{2,\ell}$

To derive $V^m_{2,b}$, take the differential of (9) with respect to $m_{2,b}$ and get

$$\begin{aligned} V^m_{2,b} &= \alpha\left[u'\left(q_{II}\right)\tfrac{\partial q_{II}}{\partial m_{2b}} + V^m_3\left(1-\tfrac{\partial z_{m,b}}{\partial m_{2,b}}\right)\right] \\ &\quad + (1-\alpha)\left[u'\left(q_I\right)\tfrac{\partial q_I}{\partial m_{2b}} + V^m_3\left(1-\tfrac{\partial z_{m,b}}{\partial m_{2,b}}\right)\right]. \end{aligned} \tag{A1}$$

Since buyers spend all their money in market 2, it holds that $\partial z_{m,b}/\partial m_{2b} = 1$, so

$$V^m_{2,b} = \alpha\left[u'\left(q_{II}\right)\frac{\partial q_{II}}{\partial m_{2b}}\right] + (1-\alpha)\left[u'\left(q_I\right)\frac{\partial q_I}{\partial m_{2b}}\right]. \tag{A2}$$

Now, take the differential of (15) and (16) with respect to $m_{2,b}$ and obtain

$$\frac{\partial q_{II}}{\partial m_{2b}} = \frac{\phi}{c'\left(q_{II}\right)} \tag{A3}$$

and

$$\frac{\partial q_I}{\partial m_{2b}} = \frac{\phi}{c'\left(q_I\right)}, \tag{A4}$$

respectively. Use $(A3)$ and $(A4)$ to eliminate $\partial q_{II}/\partial m_{2b}$ and $\partial q_I/\partial m_{2b}$ from $(A2)$, and get

$$V^m_{2,b} = \phi\left\{\alpha\frac{u'\left(q_{II}\right)}{c'\left(q_{II}\right)} + (1-\alpha)\frac{u'\left(q_I\right)}{c'\left(q_I\right)}\right\}. \tag{A5}$$

Let us now derive $V^a_{2,b}$. To do so take the differential of (9) with respect to $a_{2,b}$ and obtain

$$\begin{aligned} V^a_{2,b} &= \alpha\left[u'\left(q_{II}\right)\tfrac{\partial q_{II}}{\partial a_{2b}} + V^a_3\left(1-\tfrac{\partial z_{a,b}}{\partial m_{2,b}}\right)\right] \\ &\quad + (1-\alpha)\left[u'\left(q_I\right)\tfrac{\partial q_I}{\partial a_{2b}} + V^a_3\right]. \end{aligned} \tag{A6}$$

Again, buyers give up all their bond holdings when they trade with an informed seller, so $\partial z_{a,b}/\partial m_{2,b} = 1$. Take the differential of (15) with respect to $a_{2,b}$ and get

$$\frac{\partial q_{II}}{\partial a_{2b}} = \frac{(1+i)\,\phi}{c'\left(q_{II}\right)}. \tag{A7}$$

Note from (16) that the quantity of goods q_I exchanged in a type-I meeting does not depend upon the buyer's bond holdings, i.e. $\partial q_I/\partial a_{2b} = 0$. Hence, by virtue of (7), expression $(A6)$ can be rewritten as follows

$$V_{2,b}^{a} = \phi\left(1+i\right)\left\{\alpha\frac{u'\left(q_{II}\right)}{c'\left(q_{II}\right)} + \left(1-\alpha\right)\right\}. \tag{A8}$$

To derive $V_{2,\ell}^{m}$, take the differential of (10) with respect to $m_{2,\ell}$ and get

$$\begin{aligned} V_{2,\ell}^{m} = & \tfrac{\sigma\alpha}{1-\sigma}\left[-c'\left(q_{II}\right)\tfrac{\partial q_{II}}{\partial m_{2,\ell}} + V_3^m\left(1+\tfrac{\partial z_{m,s}}{\partial m_{2,\ell}}\right)\right] \\ & + \tfrac{\sigma(1-\alpha)}{1-\sigma}\left[-c'\left(q_I\right)\tfrac{\partial q_I}{\partial m_{2,\ell}} + V_3^m\left(1+\tfrac{\partial z_{m,s}}{\partial m_{2,\ell}}\right)\right] \\ & + \tfrac{1-2\sigma}{1-\sigma}V_3^m. \end{aligned} \tag{A9}$$

Quantities of goods (q_{II} and q_I) produced and the amount of money ($z_{m,s}$) received by a seller in a meeting depend on the buyer's money holdings, and do not depend on the seller's. Thus, it holds that $\partial q_{II}/\partial m_{2,\ell} = 0$, $\partial q_I/\partial m_{2,\ell} = 0$, $\partial z_{m,s}/\partial m_{2,\ell} = 0$, and $(A9)$ can be rewritten as

$$V_{2,\ell}^{m} = \phi \tag{A10}$$

where (7) has been used.

Let us now derive $V_{2,\ell}^{a}$ by taking the differential of (10) with respect to $a_{2,\ell}$ and get

$$\begin{aligned} V_{2,\ell}^{a} = & \tfrac{\sigma\alpha}{1-\sigma}\left[-c'\left(q_{II}\right)\tfrac{\partial q_{II}}{\partial a_{2,\ell}} + V_3^a\left(1+\tfrac{\partial z_{a,s}}{\partial a_{2,\ell}}\right)\right] \\ & + \tfrac{\sigma(1-\alpha)}{1-\sigma}\left[-c'\left(q_I\right)\tfrac{\partial q_I}{\partial a_{2,\ell}} + V_3^m\right] \\ & + \tfrac{1-2\sigma}{1-\sigma}V_3^a. \end{aligned} \tag{A11}$$

Again, terms of trade $(q_{II}, q_I, z_{a,s})$ depends on the buyer's asset holdings, and do not on the seller's, i.e. $\partial q_{II}/\partial a_{2,\ell} = 0$, $\partial q_I/\partial a_{2,\ell} = 0$, $\partial z_{a,s}/\partial a_{2,\ell} = 0$. Again, using (7), expression $(A11)$ reduces to

$$V_{2,\ell}^{a} = \phi\left(1+i\right). \tag{A12}$$

The marginal values (37)-(40) can be derived on the same lines.

Derivation of hours of work

Expected hours of work for a buyer are

$$\begin{aligned} h_b &= \alpha \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D \right\} \\ &\quad + (1-\alpha) \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,b} \right\}, \end{aligned} \tag{A13}$$

and for a nonbuyer

$$\begin{aligned} h_\ell = &\tfrac{\alpha\sigma}{1-\sigma} \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) (a_{1,\ell} + z_a) - \phi z_m + \phi \varepsilon \right\} \\ &+ \tfrac{(1-\alpha)\sigma}{1-\sigma} \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,\ell} - \phi z_m \right\} \\ &+ \tfrac{1-2\sigma}{1-\sigma} \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,\ell} \right\}. \end{aligned} \tag{A14}$$

So, aggregate hours of works in market 3 are

$$H = \sigma h_b + (1-\sigma) h_\ell \tag{A15}$$

which, using $(A13)$-$(A14)$, can be rewritten as follows

$$\begin{aligned} H = &\sigma\alpha \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D \right\} \\ &+ \sigma (1-\alpha) \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,b} \right\} \\ &+ \alpha\sigma \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) (a_{1,\ell} + z_a) - \phi z_m + \phi \varepsilon \right\} \\ &+ \sigma (1-\alpha) \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,\ell} - \phi z_m \right\} \\ &+ (1-2\sigma) \left\{ x^* + \phi m_{1,+1} + \phi T - \phi D - \phi (1+i) a_{1,\ell} \right\}, \end{aligned}$$

or, rearranging,

$$\begin{aligned} H = &x^* + \phi m_{1,+1} + \phi T - \phi D - \sigma\phi (1-\alpha) (1+i) a_{1,b} \\ &- \sigma\phi z_m - \alpha\sigma\phi (1+i) z_a - \phi (1-\sigma) (1+i) a_{1,\ell} + \phi\alpha\sigma\varepsilon. \end{aligned} \tag{A16}$$

Now, use $a_{1,b} = a_{2,b}$, $a_{1,\ell} = a_{2,\ell}$, and the fact that constraints in (11) and (12) bind, i.e. $z_m = m_{2,b}$

and $z_a = a_{2,b}$, to rewrite $(A16)$ as follows:

$$H = x^* + \phi m_{1,+1} + \phi T - \phi D - \sigma\phi(1+i)a_{2,b}$$
$$-\sigma\phi m_{2,b} - \phi(1-\sigma)(1+i)a_{2,\ell} + \phi\alpha\sigma\varepsilon,$$

which is equivalent to

$$\begin{aligned} H = \alpha x^* + \phi m_{1,+1} + \phi T - \phi D - \sigma\phi(a_{2,b} + m_{2,b}) \\ -\sigma\phi i a_{2,b} - \phi(1-\sigma)(1+i)a_{2,\ell} + \phi\alpha\sigma\varepsilon. \end{aligned} \tag{A17}$$

Note that $\alpha\sigma\varepsilon = D$ by (30), and $m_{1,+1} = a_{2,b} + m_{2,b} = a_{2,\ell} = M$, so $(A17)$ reduces to

$$H = \alpha x^* + \phi T - \phi i\left[\sigma a_{2,b} + (1-\sigma)a_{2,\ell}\right]$$

which is equivalent to

$$H = \alpha x^* + \phi T - \phi i A \tag{A18}$$

by virtue of $A = \sigma a_{2,b} + (1-\sigma)a_{2,\ell}$. Hence, using the budget constraint (2) to eliminate A from $(A18)$, and imposing symmetric condition $x^* = X^*$, aggregate hours of work in the third market are

$$H = X^*.$$