Short and Long Interest Rate Targets*

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Abstract

A target for the short-term nominal interest rate does not pin down realized inflation. Neither does it pin down the term premia. Short and long rates are threrefore independent monetary policy instruments. A target of the term structure is equivalent to a peg of the returns on state-contingent nominal assets. These are the rates that should be targeted in order to pin down realized inflation.

Key words: Monetary policy; monetary policy instruments; maturities; short rates; long rates; term structure; multiplicity of equilibria; sticky prices.

JEL classification: E3; E4; E5.

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1. Introduction

The low targets for short-term nominal interest rates during the recent financial crisis, very close to zero, prompted again the policy question of whether a central bank can target both short and long-term rates, with the hope of lowering the latter, given that the former cannot be lowered. The recent crisis has also provided empirical support for the ability of a central bank to target rates at different maturities. The Fed may have been able to influence the long-term rates, through the Quantitative Easing 1 and 2 programs in 2008-2009 and late 2010.[1] In 2009, the ECB conducted one week, three and six months, and one year, liquidity providing operations at fixed rates. And there is further historical evidence that "...a sufficiently determined Fed can peg or cap Treasury bond prices and yields at other than the shortest maturities."[2] In the 40's and 50's, before the Federal Reserve-Treasury Accord of 1951, the Fed managed to establish both the rate on the 90-day Treasury bill and a ceiling on the 12-month Treasury certificate. This was achieved without the need to hold a significant share of long maturity bonds. Operation Twist, in the 1960's, was also an attempt by the Fed to raise short rates and lower long rates.

While there is empirical evidence for the ability of a central bank to target interest rates at different maturities, there is, to our knowledge,[3] no theoretical basis for it. This is the contribution of this paper. Can both short and long-term interest rates be targeted independently? Can the target of the term structure help solve the problem of multiplicity of equilibria that occurs when only the short rate is targeted? We address both questions, and the answer is yes to both.

The problem of multiplicity of equilibria when the monetary policy instrument is the short-term, nominal interest rate was first formally addressed by Sargent and Wallace (1975). They consider an ad hoc macro model with rational expectations, assume that the policy rate responds to historical values of exogenous and endogenous variables, and show that the price level is indeterminate. Nakajima and Polemarchakis (2003) take an approach closer to ours.[4] They consider a cash-in-advance model with uncertainty and assume that policy is a target for

[1] See Hamilton and Wu (2009), D'Amico and King (2010) and Krishnamurthy and Vissing-Jorgensen (2010).

[2] From a speech by Ben Bernanke to the National Economists Club in 2002.

[3] Recent, independent work by Magill and Quinzii (2011) has similar results to ours. The focus is different: We focus on the ability of the central bank to target both short and long rates. They focus on the possibility of using the term structure to anchor expectations.

[4] See also Bloise, Dreze and Polemarchakis (2004) and Adão, Correia and Teles (2009).

the interest rate on one period nominal bonds. They compute the degrees of multiplicity. In the deterministic model there is one degree of multiplicity, say, the initial price level. Instead, when uncertainty is taken into account, there is one degree for each possible history.

This multiplicity of equilibria under uncertainty, when policy targets the short rates, is the reason for our results. It is, indeed, because there are multiple equilibrium values for the price level under a target for the short rate, that there are also multiple equilibrium values for the long-term nominal interest rates. Short and long rates are independent monetary policy instruments. A target of both short and long rates is equivalent, under general conditions, to a target of the returns on state-contingent nominal assets. If policy were to target those rates of return, it would be able to pin down the price level in every state, for a given initial price level.[5]

Sargent and Wallace (1975) and Nakajima and Polemarchakis (2003) do not consider interest rate feedback rules in which the policy rate can respond to contemporaneous endogenous variables or expectations of future variables. But considering those feedback rules does not solve the multiplicity problem. While it is possible to design Taylor-type rules such that there is a unique local equilibrium,[6] globally there are still many equilibria. The conditions for local determinacy may in fact be conditions for global indeterminacy, as shown by Benhabib, Schmitt-Grohe and Uribe (2001), Benhabib, Schmitt-Grohe and Uribe (2001, 2002, 2003) and Schmitt-Grohe and Uribe (2009), among others.[7]

Eggertsson and Woodford (2003)[8] also address the question of whether it is possible, or useful, for policy to affect long-term rates. In their model, for each equilibrium in prices and quantities, there are multiple portfolio compositions that support the equilibrium.[9] This implies that changes in the relative supply of bonds of different maturities do not affect the set of equilibria. But it does not

[5] Except for a particular implementation described in section 2.4, the initial price level remains indeterminate.

[6] Mc Callum (1981) was the first contributor to the large literature on local determinacy that followed (see Woodford (2003), among many others, and Cochrane (2007) for a critical discussion of the approach).

[7] While, in general, interest rate feedback rules give rise to multiplicity, there are cases of rules that deliver global uniqueness. This has been shown by Loisel (2008) in a linear model and by Adão, Correia and Teles (2009) in standard monetary models.

[8] See also Woodford (2005) commenting on McGough, Rudebusch, and Williams (2005). He argues that there is nothing policy on the long rates can do, that cannot be done with policy on the short rate.

[9] The argument is similar to the Modigliani-Miller, irrelevance results in Wallace (1981).

mean that the target of the prices on those assets will not affect the particular equilibrium that is implemented, as we show it does.

We start by illustrating, using a simple flexible price monetary model (section 2), that targeting the return on noncontingent short-term bonds cannot pin down the distribution of realized inflation across states (section 2.2). The nominal interest rate is a noncontingent return and therefore, although it imposes restrictions on a conditional expectation of inflation, it cannot be used to determine realized inflation. Since realized inflation is not pinned down, term premia are also not uniquely determined.

We show that the target of the short and long-term interest rates can solve the multiplicity of equilibria associated with uncertainty (section 2.3). The intuition is simple: The targeting of the term structure imposes restrictions on the term premia and therefore on the distribution of prices across states. If the number of those restrictions was equal to the number of possible contingencies, the distribution of price levels would be uniquely pinned down, for a given initial price level. In order to target the nominal term structure, the government or central bank stands ready to buy and sell any quantity of bonds of different maturities at fixed rates. We show that the net supply of those bonds can be zero in equilibrium (section 2.4).

An alternative intuition for the results uses the equivalence between completing markets with state-contingent assets and assets of different maturities (section 2.5).[10] If monetary policy were to target the returns on state-contingent nominal assets, then given an initial price level, it would be able to pin down the price level in every date and state. Targeting the returns on state-contingent nominal assets is, under certain conditions, equivalent to targeting the returns on assets of different maturities.

In section 2.6, we relate the results to the ones in the literature on local and global determinacy. And, in section 3, we extend the results to an environment with sticky prices.

2. A model with flexible prices

We consider first a simple cash-in-advance economy with flexible prices. The economy consists of a representative household, a representative firm behaving

[10] The results are related to the ones in Angeletos (2002) and Buera and Nicolini (2004) that have shown that state-contingent public debt may be replicated with public debt of multiple maturities.

competitively, and a government. The uncertainty in period $t \geq 0$ is described by the random variable $s_t \in S_t$, where S_t is the set of possible events at t, and the history of its realizations up to period t (state or node at t), $(s_0, s_1, ..., s_t)$, is denoted by $s^t \in S^t$. We assume that s_t has a discrete distribution. The number of states in period $t \geq 0$ is Φ_t. There is fundamental uncertainty if technology and government spending $A(s^t)$ and $G(s^t)$ are functions of the state s^t. Otherwise uncertainty is nonfundamental.

Production uses labor according to a linear technology. We impose a cash-in-advance constraint on the households' transactions with the timing structure described in Lucas and Stokey (1983). That is, each period is divided into two subperiods, with the assets market operational in the first subperiod and the goods market in the second.

2.1. Competitive equilibria

Households The households have preferences over consumption $C(s^t)$, and leisure $L(s^t)$, described by the expected utility function

$$U = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u\left(C\left(s^t\right), L\left(s^t\right)\right) \right\} \tag{2.1}$$

where E_t is the expectation conditional on the information in state s^t and β is a discount factor. The households start period t, in state s^t, with nominal wealth $\mathbb{W}(s^t)$. They decide to hold money, $M(s^t)$, and to buy $B^j(s^t)$ nominal bonds that pay $R^j(s^t) B^j(s^t)$, $j = 1, ..., m$, periods later. $R^1(s^t)$ is the gross short-term nominal interest rate. They also buy $Z(s^{t+1})$ units of state-contingent nominal securities. Each security pays one unit of money at the beginning of period $t+1$ in state s^{t+1}. Let $Q(s^{t+1}/s^t)$ be the beginning of period t price of these securities normalized by the probability of the occurrence of the state. The households spend $E_t Q(s^{t+1}/s^t) Z(s^{t+1})$ in state-contingent nominal securities. Thus, in the assets market at the beginning of period t they face the constraint

$$M\left(s^t\right) + \sum_{j=1}^{m} B^j\left(s^t\right) + E_t Q\left(s^{t+1}/s^t\right) Z\left(s^{t+1}\right) \leq \mathbb{W}\left(s^t\right) \tag{2.2}$$

where the initial nominal wealth $\mathbb{W}(s_0)$ is given.

Consumption must be purchased with money according to the cash-in-advance constraint

$$P\left(s^t\right) C\left(s^t\right) \leq M\left(s^t\right). \tag{2.3}$$

At the end of the period, the households receive the labor income $W\left(s^t\right) N\left(s^t\right)$, where $N\left(s^t\right) = 1 - L\left(s^t\right)$ is labor and $W\left(s^t\right)$ is the nominal wage rate and pay lump sum taxes $T\left(s^t\right)$. Thus, the nominal wealth households bring to state s^{t+1} is

$$\begin{aligned} \mathbb{W}\left(s^{t+1}\right) &= M\left(s^t\right) + \sum_{j=1}^{m} R^j\left(s^{t+1-j}\right) B\left(s^{t+1-j}\right) + Z\left(s^{t+1}\right) \\ &\quad - P\left(s^t\right) C\left(s^t\right) + W\left(s^t\right) N\left(s^t\right) - T\left(s^t\right) \end{aligned} \tag{2.4}$$

The households' problem is to maximize expected utility (2.1) subject to the restrictions (2.2), (2.3), (2.4), together with a no-Ponzi games condition on the holdings of assets.

The following are first order conditions of the households' problem:

$$\frac{u_L\left(s^t\right)}{u_C\left(s^t\right)} = \frac{W\left(s^t\right)}{P\left(s^t\right) R^1\left(s^t\right)}, \tag{2.5}$$

$$\frac{u_C\left(s^t\right)}{P\left(s^t\right)} = R^j\left(s^t\right) E_t\left[\frac{\beta^j u_C\left(s^{t+j}\right)}{P\left(s^{t+j}\right)}\right],\ j = 1, ..., m, \tag{2.6}$$

$$Q\left(s^{t+1}/s^t\right) = \beta \frac{u_C\left(s^{t+1}\right)}{u_C\left(s^t\right)} \frac{P\left(s^t\right)}{P\left(s^{t+1}\right)}. \tag{2.7}$$

From these conditions we get

$$E_t Q\left(s^{t+1}/s^t\right) = \frac{1}{R^1\left(s^t\right)}. \tag{2.8}$$

Condition (2.5) sets the intratemporal marginal rate of substitution between leisure and consumption equal to the real wage adjusted for the cost of using money, $R^1\left(s^t\right)$. Condition (2.6) is an intertemporal marginal condition necessary for the optimal choice of risk-free nominal bonds of different maturities. Condition (2.7) equates the price of one unit of money at time $t+1$, for each state s^{t+1}, in units of money at time t, in state s^t,[11] to the intertemporal marginal rate of substitution.

[11] $Q\left(s^{t+1}/s^t\right)$ is this price normalized by the conditional probability of occurrence of state s^{t+1}.

Firms The firms are competitive and prices are flexible. The production function of the representative firm is linear

$$Y\left(s^t\right) = A\left(s^t\right) N\left(s^t\right)$$

The equilibrium real wage is

$$\frac{W\left(s^t\right)}{P\left(s^t\right)} = A\left(s^t\right). \tag{2.9}$$

Government The policy variables are taxes $T\left(s^t\right)$, nominal interest rates $R^j\left(s^t\right)$, state-contingent nominal prices $Q\left(s^{t+1}/s^t\right)$, money supplies $M\left(s^t\right)$, state-noncontingent public debt $B^j\left(s^t\right)$ and state-contingent debt $Z\left(s^{t+1}\right)$. The government expenditures, $G\left(s^t\right)$, are exogenous.

The government budget constraints are

$$\begin{aligned} M\left(s^t\right) + \sum_{j=1}^{m} B^j\left(s^t\right) + E_t Q\left(s^{t+1}/s^t\right) Z\left(s^{t+1}\right) &= \\ M\left(s^{t-1}\right) + \sum_{j=1}^{m} R^j\left(s^{t-j}\right) B\left(s^{t-j}\right) + Z\left(s^t\right) + P\left(s^{t-1}\right) G\left(s^{t-1}\right) - T\left(s^{t-1}\right), t &\geq 0 \end{aligned}$$

together with a no-Ponzi games condition. Let $Q\left(s^{t+1}\right) \equiv Q\left(s^{t+1}/s_0\right)$, with $Q\left(s_0\right) = 1$. If $\lim_{T\rightarrow\infty} E_t Q\left(s^{T+1}\right) \mathbb{W}\left(s^{T+1}\right) = 0$, the budget constraints can be written as

$$\begin{aligned} & \sum_{s=0}^{\infty} E_t Q\left(s^{t+s+1}/s^t\right)\left[M\left(s^{t+s}\right)\left(R^1\left(s^{t+s}\right)-1\right) + T\left(s^{t+s}\right) - P\left(s^{t+s}\right) G\left(s^{t+s}\right)\right] \\ = \; & \mathbb{W}\left(s^t\right) \end{aligned} \tag{2.10}$$

Market clearing The goods and labor market clearing conditions are

$$C\left(s^t\right) + G\left(s^t\right) = A\left(s^t\right) N\left(s^t\right)$$

and

$$1 - L\left(s^t\right) = N\left(s^t\right).$$

We have already imposed market clearing in the money and asset markets.

Equilibria The competitive equilibrium conditions for the variables $\{C(s^t), L(s^t)\}$, and $\{P(s^t), R^j(s^t), Q(s^{t+1}/s^t), M(s^t), B^j(s^t), Z(s^{t+1}), T(s^t)\}$, with $R^j(s^t) \geq 1$, are the resource constraints

$$C\left(s^t\right) + G\left(s^t\right) = A\left(s^t\right)\left(1 - L\left(s^t\right)\right), \tag{2.11}$$

the intratemporal conditions

$$\frac{u_C(s^t)}{u_L(s^t)} = \frac{R^1(s^t)}{A(s^t)}, \tag{2.12}$$

obtained from the households intratemporal conditions (2.5) and the firms optimal condition (2.9), as well as the cash-in-advance constraints (2.3), the intertemporal conditions (2.6) and (2.7), and the budget constraints (2.10). Notice that, given the nominal interest rate $R^1(s^t)$, the consumption and leisure allocation in each state s^t, $C(s^t)$ and $L(s^t)$, is uniquely determined by the resource constraint (2.11) and the intratemporal condition (2.12), for each state s^t.

The equations identified above determine a set of equilibrium allocations, prices and policy variables. In order for a particular equilibrium in this set to be implemented, it is necessary to determine exogenous policy rules for a subset of the policy variables.

2.2. Multiplicity of equilibria with a target for the short rate

When monetary policy is a target for the short-term nominal interest rate the initial price level is not pinned down.[12] But, even if the initial price level was uniquely determined, the distribution of prices in the subsequent periods would not be unique, when uncertainty, whether fundamental or not, is taken into account. The multiplicity of the distribution of realized inflation across states implies that, in general, the term premia are not pinned down and therefore the term structure is also not pinned down. It follows that interest rates of different maturities are independent policy instruments.

We start by showing that if the policy instrument is the short-term nominal interest rate, given an initial price level, it is possible to implement a unique equilibrium in the deterministic economy, but not under uncertainty. From the resource constraints, (2.11) and the intratemporal conditions (2.12), we can write

[12] This is the indeterminacy - of the initial price level - that most of the literature focuses on. Most of the analysis is in deterministic models, and there, for a given initial price level, there is a single path for the price level in subsequent periods.

consumption and leisure as functions of the short-term nominal rate, $C(R^1(s^t))$ and $L(R^1(s^t))$. Let $u_C(R^1(s^t)) \equiv u_C(C(R^1(s^t)), L(R^1(s^t)))$. Suppose the short-term nominal interest rate $R^1(s^t)$ is set exogenously in every date and state. The allocation is then pinned down uniquely. The issue is how can a unique sequence of prices levels be pinned down.

The path for the price level $P(s^t)$ is restricted by the following dynamic equations, only:

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta R^1\left(s^t\right) E_t\left[\frac{u_C(R^1(s^{t+1}))}{P(s^{t+1})}\right], \, t \geq 0. \tag{2.13}$$

The other equations restrict the other variables: Given a process for the price level $P(s^t)$, the intertemporal conditions (2.6) for $j = 2, ..., m$, and (2.7), determine, respectively $R^j(s^t)$ for $j = 2, ..., m$, and $Q(s^{t+1}/s^t)$, the cash-in-advance constraints, (2.3), restrict $M(s^t)$, the budget constraints, (2.10), can be satisfied by the choice of lump sum taxes $T(s^t)$.

If there was no uncertainty, given an initial price level, the intertemporal conditions (2.13) would determine the price level uniquely for every date $t \geq 1$. Instead, under uncertainty, even if the initial price level is given, there are still multiple equilibria for the price level in each state. To see this, notice that in any period $t \geq 1$, given $P(s^{t-1})$, there are Φ_{t-1} equations to determine Φ_t variables, $P(s^t)$, where, again, Φ_t is the number of states at t. More specifically, for each state s^{t-1}, there is one equation to determine $\#S_t$ variables, where $\#S_t$ is the number of possible events at t. Except for the deterministic case, there are multiple solutions for the price level $P(s^t)$.

The indeterminacy of the initial price level in the deterministic economy becomes the indeterminacy of one price level per history, under uncertainty. As we will see below this explosion in the degrees of multiplicity under uncertainty results from pegging the noncontingent nominal interest rate instead of the returns on state-contingent nominal assets. If, instead, these were pegged, there would be a single degree of multiplicity as in the deterministic case.

Notice that arbitrage between one period and two period maturity assets, from (2.6) for $j = 1, 2$, can be written as[13]

$$\frac{R^1(s^t)}{R^2(s^t)} = E_t\left[\frac{1}{R^1(s^{t+1})}\right] + \frac{Cov_t\left(\frac{1}{R^1(s^{t+1})}, \frac{u_C\left(R^1\left(s^{t+1}\right)\right)}{P(s^{t+1})}\right)}{E_t\left[\frac{u_C(R^1(s^{t+1}))}{P(s^{t+1})}\right]}. \tag{2.14}$$

[13] The derivation of the condition for $j = 2, ..., m$, is in the Appendix A.2.

The multiplicity of distributions of price levels when policy is a target for $R^1(s^t)$, for all t, implies that the term premium, measured by the covariance term in (2.14), is also, in general, not pinned down and therefore there are degrees of freedom in determining the returns on assets of longer maturities, $R^2(s^t)$. Similarly, the term premia for longer maturities are also not pinned down.

To determine the degrees of freedom in targeting long-term rates, we show next how those rates can be targeted to implement a unique equilibrium, for a given initial price level.

2.3. Targeting the term structure

We derive in this section the main results in the paper. In order to illustrate the results, we consider that in each period there are two possible events, $s_t \in \{h, l\}$, $t \geq 1$, and suppose that there are one and two period noncontingent bonds. Let $\pi(s_{t+1}/s^t)$ be the probability of s_{t+1} conditional on state s^t. Then the following conditions must be satisfied:

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta R^1(s^t)\left[\pi\left(h/s^t\right)\frac{u_C(R^1(s^t,h))}{P(s^t,h)} + \pi\left(l/s^t\right)\frac{u_C(R^1(s^t,l))}{P(s^t,l)}\right] \tag{2.15}$$

$$\begin{aligned} &\frac{u_C(R^1(s^t))}{P(s^t)} \\ = \;&\beta^2 R^2\left(s^t\right)\left[\begin{array}{c} \pi\left((h,h)/s^t\right)\frac{u_C\left(R^1\left(s^t,h,h\right)\right)}{P(s^t,h,h)} + \pi\left((h,l)/s^t\right)\frac{u_C\left(R^1\left(s^t,h,l\right)\right)}{P(s^t,h,l)} \\ +\pi\left((l,h)/s^t\right)\frac{u_C\left(R^1\left(s^t,l,h\right)\right)}{P(s^t,l,h)} + \pi\left((l,l)/s^t\right)\frac{u_C\left(R^1\left(s^t,l,l\right)\right)}{P(s^t,l,l)} \end{array}\right] \end{aligned} \tag{2.16}$$

$$\begin{aligned} &\frac{u_C(R^1(s^t,h))}{P(s^t,h)} \\ = \;&\beta R^1\left(s^t,h\right)\left[\pi\left(h/\left(s^t,h\right)\right)\frac{u_C(R^1(s^t,h,h))}{P(s^t,h,h)} + \pi\left(l/\left(s^t,h\right)\right)\frac{u_C(R^1(s^t,h,l))}{P(s^t,h,l)}\right] \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\frac{u_C(R^1(s^t,l))}{P(s^t,l)} \\ = \;&\beta R^1\left(s^t,l\right)\left[\pi\left(h/\left(s^t,l\right)\right)\frac{u_C(R^1(s^t,l,h))}{P(s^t,l,h)} + \pi\left(l/\left(s^t,l\right)\right)\frac{u_C(R^1(s^t,l,l))}{P(s^t,l,l)}\right] \end{aligned} \tag{2.18}$$

The last three conditions, (2.16), (2.17), and (2.18), can be used to obtain

$$\begin{aligned} & \frac{u_C\left(R^1\left(s^t\right)\right)}{P\left(s^t\right)} \\ = \quad & \beta R^2\left(s^t\right)\left[\pi\left(h/s^t\right)\frac{u_C\left(R^1\left(s^t,h\right)\right)}{R^1\left(s^t,h\right)P\left(s^t,h\right)}+\pi\left(l/s^t\right)\frac{u_C\left(R^1\left(s^t,l\right)\right)}{R^1\left(s^t,l\right)P\left(s^t,l\right)}\right] \end{aligned} \tag{2.19}$$

Suppose policy is a target for the interest rates on the one and two-period-maturity bonds, $R^1\left(s^t\right)$ and $R^2\left(s^t\right)$. Given $P\left(s^t\right)$, conditions (2.15) and (2.19) determine $P\left(s^t,h\right)$ and $P\left(s^t,l\right)$, provided $R^1\left(s^t,h\right) \neq R^1\left(s^t,l\right)$. It follows that if $R^1\left(s^t\right) \geq 1$ and $R^2\left(s^t\right) \geq 1$ are set exogenously, and $R^1\left(s^t,h\right) \neq R^1\left(s^t,l\right)$, for all s^t, for a given initial price level $P\left(s_0\right)$, there is a unique solution for the allocations and prices.

Conditions (2.15) and (2.19), are two linear equations in two unknowns, the inverse of the price levels in the two states. As long as the matrix of the coefficients of $\frac{1}{P(s^t,l)}$ and $\frac{1}{P(s^t,h)}$ is invertible, there is a single solution. In order for that to be the case it must be that the two rates $R^1\left(s^t,h\right)$ and $R^1\left(s^t,l\right)$ are different, but they can differ by an arbitrarily small number.[14] In this sense, the conditions for the invertibility of the matrix of coefficients are general.

For the general case with n contingencies and m maturities, m cannot be lower than n. n maturities must be pegged independently in a way that guarantees that the nxn matrix of coefficients described above is invertible. This holds generally. The proposition follows.

Proposition 2.1. *Let $S_t = \{s_1, s_2, ..., s_n\}$ and suppose there are nominal non-contingent assets of maturity $j = 1, ..., m$. Let $m \geq n$. If the returns on n of these assets are set exogenously, then, in general, there is a unique equilibrium for the allocations and prices, given the initial price level $P\left(s_0\right)$.*

Proof: See Appendix A.1.■

As n becomes arbitrarily large, the whole term structure can be pegged. This is the main result of the paper, stated in the following corollary.

Corollary 2.2. *Short and long-term nominal interest rates are independent monetary policy instruments.*

[14] If the variability of the short rates was very low, the determinant of the matrix of coefficients of the price levels would be close to zero. This would mean that small changes in the targets for the longer rates, would have a large effect on the values for the price levels. We thank Marco Basseto for pointing this out to us.

While the question of using long rates as an instrument of policy is typically raised when short rates are very close to the zero lower bound, it turns out that, at the zero bound, the conditions for the result in proposition 2.1 may not be verified.

If the economy was always at the zero bound, in every period and state, $R^1(s^t) = 1$ for all t and s^t, then the covariance in the arbitrage condition (2.14), would be zero for any process of the price level. This means that targeting the longer rates does not impose restrictions on the price level, and it also means that the longer rates are uniquely obtained from the short rates. Indeed, from (2.14), the expectations hypothesis would trivially hold, and $R^j(s^t) = 1$ for all j and s^t. The condition of enough variability in the interest rates is not fulfilled. Short and long rates would not be independent policy instruments and the process for the price level would not be pinned down.

An ε deviation from the zero bound would, however, allow to recover the results. Notice also that, if the economy was temporarily at the zero bound, as is normally expected to be the case, then it would still be possible to pin down price levels, and short and longer maturities would still be independent targets. The returns on shorter maturities would be obtained from the even shorter rates, but longer rates could still be targeted independently. To see this, suppose, for instance, that the economy was at the zero bound in periods t and $t+1$ for all possible contingencies, meaning that $R^1(s^t) = R^1(s^t, l) = R^1(s^t, h) = 1$. Then, from (2.15) and (2.19), the two period return in period t would be given by

$$R^2\left(s^t\right) = R^1\left(s^t\right) R^1\left(s^{t+1}\right) = 1$$

and the two period rate would be obtained using the short rates, so that it would not be a separate instrument. But suppose there was a target for the returns on the one and three-period bonds, $R^1(s^t)$ and $R^3(s^t)$. For three-period maturity bonds, we would have

$$\begin{aligned} & \frac{u_C\left(R^1\left(s^t\right)\right)}{P\left(s^t\right)} \\ = \quad & \beta R^3\left(s^t\right)\left[\pi\left(h/s^t\right)\frac{u_C\left(R^1\left(s^t,h\right)\right)}{R^2\left(s^t,h\right)P\left(s^t,h\right)} + \pi\left(l/s^t\right)\frac{u_C\left(R^1\left(s^t,l\right)\right)}{R^2\left(s^t,l\right)P\left(s^t,l\right)}\right]. \end{aligned} \tag{2.20}$$

If $R^2(s^t, h) \neq R^2(s^t, l) \neq 1$, we could use (2.15) and (2.20) to determine the price levels $P(s^t, h)$ and $P(s^t, l)$. In this case, the three period return is an independent target.

2.4. Debt in zero net supply

In targeting the interest rates on the assets of different maturities, the government, or central bank, stands ready to supply and demand any quantity of bonds of different maturities, at given rates. In our model, where we assume that there are lump sum taxes, the net supply of those assets is not uniquely determined. In particular, it can be zero in equilibrium.

The conditions restricting the supply of the noncontingent bonds are the government budget constraints

$$\begin{aligned}
&\sum_{s=0}^{\infty} E_t Q\left(s^{t+s+1}/s^t\right)\left[M\left(s^{t+s}\right)\left(R^1\left(s^{t+s}\right)-1\right)+T\left(s^{t+s}\right)-P\left(s^{t+s}\right)G\left(s^{t+s}\right)\right] \\
= \;& \mathbb{W}\left(s^t\right),\ t=1,2...
\end{aligned}$$

where $\mathbb{W}\left(s^t\right) = M\left(s^{t-1}\right) + Z\left(s^t\right) + \sum_{j=1}^{m} R^j\left(s^{t-j}\right) B\left(s^{t-j}\right) + P\left(s^{t-1}\right) G\left(s^{t-1}\right) - T\left(s^{t-1}\right)$.[15] But these conditions also restrict the path for the taxes, as well as the supply of the state-contingent debt.

Suppose, first, that the government cannot issue state-contingent debt, so that $Z\left(s^t\right) = 0$. And, furthermore, suppose that taxes in every state s^t are given exogenously. Consider the conditions in Proposition 2.1 that guarantee that there is a unique equilibrium, given the initial price level. Then there is also a unique solution for the supply of the assets of the n maturities. Notice also that in this case the initial price level is pinned down by the budget constraint in period 0, according to the fiscal theory of the price level. The subsequent price levels are not pinned down by the budget constraints for each state, since those are satisfied by the supply of debt of different maturities, that replicates state-contingent debt. Those price levels are pinned down by the target of the term structure.

If, instead, lump sum taxes, as we assume, are endogenous, then, the supply of the noncontingent of different maturities is not uniquely determined. Suppose, in particular, that the supply of the assets of different maturities is set to zero, $B\left(s^{t-j}\right) = 0$, $j = 1, ..., m$. It is still possible to use the lump sum taxes, $T\left(s^{t-1}\right)$, to generate the same nominal liabilities of the government in each state s^t, $t \geq 1$.

In this economy, while the direct target of the returns on the noncontingent assets of different maturities can implement a unique equilibrium for the allocation and price levels, there are still multiple portfolio compositions supporting that

[15] $\mathbb{W}\left(s_0\right)$ is exogenous.

equilibrium. The asset composition must replicate the outstanding nominal liabilities of the government and there are many alternative ways of accomplishing it, using lump sum taxes, supply of bonds of different maturities or state-contingent debt. In order for the supply of noncontingent bonds of different maturities to be uniquely determined, there must be restrictions imposed on taxes and state-contingent debt. This result is in line with the one identified by Eggertsson and Woodford (2003) or, more generally, with the irrelevance results in Wallace (1981).

2.5. Targeting the returns on state-contingent nominal assets

In these monetary economies where policy is a target for the interest rate on a noncontingent nominal asset, expected inflation is pinned down, but realized inflation is not. But what if, instead, policy was a target for the return on state-contingent nominal assets? Would realized inflation be pinned down for each contingency? That is indeed the case. If the central bank was to target the returns on state-contingent assets, for every contingency, then, given an initial price level, it would be able to pin down the price level for any contingency. This result is not surprising, at this stage, since targeting the term structure is an alternative way to target those state-contingent returns.

That the number of maturities has to be larger than the number of contingencies is a necessary condition. The other conditions are the ones identified above on the variability of the returns on the noncontingent assets at different maturities. This result is similar to the one obtained in Angeletos (2002) and Buera and Nicolini (2004) on replicating state-contingent public debt with debt of different maturities. Their emphasis is on quantities, ours on prices, but the mechanisms are similar.

To show the result that a target of the state-contingent prices $Q\left(s^t/s^{t-1}\right)$, satisfying $R^j\left(s^t\right) = \frac{1}{E_t[Q(s^{t+1}/s^t)Q(s^{t+2}/s^{t+1})...Q(s^{t+m}/s^{t+m-1})]} \geq 1$, $j = 1, ..., m$, determines the price level $P\left(s^t\right)$ for $t \geq 1$, given the initial price $P\left(s^0\right)$, notice that the equilibrium conditions restricting $Q\left(s^t/s^{t-1}\right)$, $R^1\left(s^t\right)$ and $P\left(s^t\right)$ can be summarized by the dynamic equations (2.7) and (2.8), repeated here:

$$Q\left(s^t/s^{t-1}\right) = \beta \frac{u_C(R^1\left(s^t\right))}{u_C(R^1\left(s^{t-1}\right))} \frac{P\left(s^{t-1}\right)}{P\left(s^t\right)}, \ t \geq 1 \tag{2.21}$$

and

$$E_t Q\left(s^{t+1}/s^t\right) = \frac{1}{R^1\left(s^t\right)}.$$

The other equations restrict the other variables: The intertemporal conditions (2.6) for $j = 2, ..., m$, determine $R^j(s^t)$ also , the cash-in-advance constraints, (2.3), restrict $M(s^t)$, and the budget constraints, (2.10), can be satisfied by the choice of lump sum taxes $T(s^t)$.

Clearly if policy is conducted by setting exogenously the state-contingent nominal returns, given the initial price level, the price levels, for every date and state, are all determined. The proposition follows:

Proposition 2.3. *If the returns on one-period, state-contingent, nominal assets are set exogenously for every date and state, given an initial price level, there is a unique equilibrium for the allocations and prices.*

Proof: Let $P(s_0)$ be given. Given the values for $Q(s^t/s^{t-1})$, $t \geq 1$, $R^1(s^t)$, $t \geq 0$ are determined uniquely, and given $P(s^{t-1})$, $P(s^t)$ is obtained from the intertemporal conditions (2.21) for $t \geq 1$. The conditions only hold for $t \geq 1$. Cannot use the condition at $t = 0$, to determine $P(s_0)$.[16] ■

2.6. The pure expectations hypothesis and local determinacy

In this section we further interpret the results and compare them to the results in the literature on local and global determinacy.

Arbitrage between holding a two period bond to maturity and rolling over one period bonds for two periods implies

$$R^2\left(s^t\right) E_t\left[\frac{u_C(R^1\left(s^{t+2}\right))}{P\left(s^{t+2}\right)}\right] = R^1\left(s^t\right) E_t\left[R^1\left(s^{t+1}\right) E_{t+1}\left[\frac{u_C(R^1\left(s^{t+2}\right))}{P\left(s^{t+2}\right)}\right]\right], \tag{2.22}$$

which is implied by (2.6) for $j = 1, 2$. If the covariance between $R^1(s^{t+1})$ and $E_{t+1}\left[\frac{u_C(R^1(s^{t+2}))}{P(s^{t+2})}\right]$ was always zero, for any process of the price level $P(s^{t+2})$, then

$$R^2\left(s^t\right) = R^1\left(s^t\right) E_t\left[R^1\left(s^{t+1}\right)\right], \tag{2.23}$$

which is the pure expectations hypothesis of the term structure. In this case, targeting the short rates would also set the long rates. Clearly short and long rates would not be independent instruments. Policy on the long rates could not be used to determine the price level, that would stay indeterminate.

[16] $M(s_0)$ could pin down the initial price from the cash-in-advance constraint that, if $R^1(s_0) > 1$, holds with equality.

But, in general, the covariance between $R^1\left(s^{t+1}\right)$ and $E_{t+1}\left[\frac{u_C\left(R^1\left(s^{t+2}\right)\right)}{P\left(s^{t+2}\right)}\right]$ is not zero. And it can be a function of policy. Targeting both short and long rates, pins down the covariance, or the term premium. Those are additional restrictions on the process for the price level, that allow to implement uniquely a desirable process for allocations and prices (again, given an initial price level).

It is important to note, that the loglinearization of the equilibrium conditions, sets the term premia to zero, and therefore the expectations hypothesis is always verified. The loglinearization of the left hand side of the arbitrage condition (2.22) is $\widehat{R^2\left(s^t\right)} + \frac{u_{CR}R^1}{u_C}E_t\widehat{R^1\left(s^{t+2}\right)} - E_t\widehat{P\left(s^{t+2}\right)}$ and the one of the right hand side is $\widehat{R^1\left(s^t\right)} + E_t\widehat{R^1\left(s^{t+1}\right)} + \frac{u_{CR}R^1}{u_C}E_t\widehat{R^1\left(s^{t+2}\right)} - E_t\widehat{P\left(s^{t+2}\right)}$ implying

$$\widehat{R^2\left(s^t\right)} = \widehat{R^1\left(s^t\right)} + E_t\widehat{R^1\left(s^{t+1}\right)},$$

where, for each variable x_t, we define $\widehat{x}_t = \log x_t - \log x$, where x is the steady-state value for the variable x_t. Also here, in the linearized model, short and long rates are not independent instruments and it is not possible to restrict term premia, because they are zero by assumption. Prices are indeterminate.

In an attempt to side step the multiplicity problem, McCallum (1981) proposed an interest rate feedback rule such that there is a locally determinate equilibrium[17] at the expense of multiple explosive solutions. Our approach is different from this more common approach to implementation (see Woodford, 2005) for two reasons: First, and foremost, we do not loglinearize, and therefore term premia are not automatically set to zero. But there is also a conceptual difference. It would not be possible, nor necessary, to use both short and long rates, if the locally determinate equilibrium was indeed the single equilibrium. But it is not. The locally determinate equilibrium is only one of the possible equilibria, as stressed by the literature on global multiplicity, as in Benhabib, Schmitt–Grohe and Uribe (2001b), Benhabib, Schmitt-Grohe and Uribe (2001b, 2002, 2003) or Schmitt-Grohe and Uribe (2009).

[17] This means that the linear system of equations that approximates the equilibrium conditions in the neighborhood of a steady state, has a unique solution in that neighborhood and multiple solutions outside that neighborhood.

3. Price setting restrictions

In a flexible price economy, when policy is conducted with interest rate targets, prices are not pinned down but allocations are. Instead, under sticky prices, setting the path for the nominal interest rates not only does not pin down prices, it also generates multiple equilibria in the allocations. In this section we show that the results derived above extend to an environment with sticky prices.

We modify the environment to consider price setting restrictions. There is, now, a continuum of goods, indexed by $i \in [0,1]$. Each good i is produced by a different firm. The firms are monopolistic competitive and set prices one period in advance. The results can be easily extended to the case where firms set prices in advance with different lags, to allow for price dispersion (see Adão, Correia and Teles, 2010).

The households have preferences described by (2.5) where $C(s^t)$ is now the standard Dixit-Stiglitz aggregator of the consumption of the individual $i \in [0,1]$ goods, $c_i(s^t)$. The households' intertemporal and intratemporal conditions on the aggregates are as before, (2.5), (2.6) and (2.7).

The government must finance an exogenous path of government purchases $\{G(s^t)\}_{t=0}^{\infty}$, which is also a Dixit-Stiglitz aggregator of individual $g_i(s^t)$, with the same elasticity of substitution as for private consumption, θ.

Each firm i chooses the price $p_i(s^t)$ for period t with the information up to period $t-1$ to maximize profits

$$E_{t-1}\left[Q\left(s^{t+1}/s^{t-1}\right)\left(p_i\left(s^t\right)y_i\left(s^t\right) - W\left(s^t\right)n_i\left(s^t\right)\right)\right]$$

subject to the production function

$$y_i\left(s^t\right) \leq A_t n_i\left(s^t\right)$$

and the demand function,

$$y_i\left(s^t\right) = \left(\frac{p_i\left(s^t\right)}{P\left(s^t\right)}\right)^{-\theta} Y\left(s^t\right) \tag{3.1}$$

derived from expenditure minimization by households and government, where $y_i(s^t) = c_i(s^t) + g_i(s^t)$. $P(s^t)$ is the aggregate price level.

The optimal price, common to all firms, is

$$p_i\left(s^t\right) = P\left(s^t\right) = \frac{\theta}{(\theta-1)}E_{t-1}\left[\eta\left(s^{t+1}\right)\frac{W\left(s^t\right)}{A\left(s^t\right)}\right] \tag{3.2}$$

where $\eta\left(s^{t+1}\right)=\frac{Q\left(s^{t+1}/s^{t-1}\right)P\left(s^{t}\right)^{\theta}Y\left(s^{t}\right)}{E_{t-1}\left[Q\left(s^{t+1}/s^{t-1}\right)P\left(s^{t}\right)^{\theta}Y\left(s^{t}\right)\right]}$.

Substituting the state-contingent prices $Q\left(s^{t+1}/s^{t-1}\right)$ in the price setting conditions (3.2), and using the intertemporal condition (2.6) as well as the households' intratemporal condition (2.5), we can write the intratemporal conditions

$$E_{t-1}\left[\frac{u_C\left(s^t\right)}{R^1\left(s^t\right)}A\left(s^t\right)\left(1-L\left(s^t\right)\right)-\frac{\theta}{(\theta-1)}u_L\left(s^t\right)\left(1-L\left(s^t\right)\right)\right]=0,\ t\geq 1. \quad (3.3)$$

Under flexible prices there was one such condition per state s^t, (2.12). That condition and the resource constraint (2.11) determined the allocation $C\left(s^t\right)$ and $L\left(s^t\right)$ as a function of the short rate $R^1\left(s^t\right)$. Not so under sticky prices. Under sticky prices, allocations are not pinned down by the path of the short-term nominal interest rates.

The equilibrium conditions for $\left\{C\left(s^t\right), L\left(s^t\right), R^j\left(s^t\right), P\left(s^t\right)\right\}$ can be summarized by the intertemporal conditions (2.6), the intratemporal conditions (3.3), and the resource constraints (2.11). Again, the other conditions restrict the other variables.

Compared with the equilibrium conditions under flexible prices for the variables $\left\{C\left(s^t\right), L\left(s^t\right), R^j\left(s^t\right), P\left(s^t\right)\right\}$, there are less conditions here, but there are also less variables to determine. The intratemporal conditions (2.12), one for each state, are replaced by (3.3), which are as many as the number of states in the previous period. But the price levels, one per state under flexible prices are also replaced by the predetermined prices under sticky prices. The number of variables and restrictions is the same.

When policy is a target for the short rate, the degree of multiplicity is the same as under flexible prices. The result in proposition 2.1, that a peg of the term structure delivers a unique equilibrium, for a given initial price, still holds when prices are set in advance.

The proposition, proved in the Appendix A.3, follows:

Proposition 3.1. *Suppose prices are set in advance. Let* $S_t=\{s_1, s_2, ..., s_n\}$ *and suppose there are nominal noncontingent assets of maturity* $j=1,...,m$. *Let* $m\geq n$. *If the returns on* n *of these assets are set exogenously, then, in general, there is a unique equilibrium for the allocations and prices, given an initial level of consumption* $C\left(s_0\right)$.

Also in this environment targeting the term structure of interest rates is equivalent to targeting the returns on state-contingent nominal assets. Notice that the number of intertemporal conditions for the state-contingent assets (2.7) is the same as the number of intertemporal conditions for noncontingent assets of different maturities (2.6), when the number of maturities is equal to the number of events.

In this case with all prices set one period in advance, given the price level $P(s^t)$ that is predetermined, a target for the state-contingent prices $Q(s^t/s^{t-1})$, allows to pin down the allocation $C(s^t)$, $L(s^t)$, from

$$Q\left(s^t/s^{t-1}\right) = \beta \frac{u_C(C(s^t), L(s^t))}{u_C(C(s^{t-1}), L(s^{t-1}))} \frac{P(s^{t-1})}{P(s^t)},\ t \geq 1,$$

and

$$C\left(s^t\right) + G\left(s^t\right) = A\left(s^t\right) N\left(s^t\right).$$

4. Concluding Remarks

We make two main points in this paper. The first result of practical interest for policy, is that a central bank can independently target both short and long-term nominal interest rates, possibly the whole term structure. This helps explain the apparent ability of central banks to peg interest rates at different maturities; being the operations of the Fed and the ECB during the recent financial crisis, of 2008 and 2009, the most striking evidence of it.

The second result is weaker and is of mostly theoretical interest. We show that setting both short and long-term nominal interest rates allows to solve the problem of multiplicity of equilibria associated with uncertainty, that arises when monetary policy is conducted with an interest rate rule for the noncontingent, short-term, nominal interest rate. The result is general in the context of the model. But, it relies on assumptions on number of contingencies and maturities that are hard to translate into real data. A necessary condition is that the number of maturities that are independently targeted equals the number of possible contingencies. But what are the relevant contingencies in the actual economy? Is there a continuum of those?

The target of the term structure is equivalent, under general conditions, to the target of the state-contingent nominal returns. A monetary policy that targets those returns is also able to implement a unique equilibrium, for a given initial price level. But, again, is it reasonable to think that the central bank is able to target the prices on state-contingent nominal assets for any possible contingency, including nonfundamental uncertainty?

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A. Appendix

A.1. Proposition 2.1

Let $S_t = \{s_1, s_2, ..., s_n\}$. Then it must be the case that

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta R^1\left(s^t\right)\left[\begin{array}{c} \pi\left(s_1|s^t\right)\frac{u_C\left(R^1\left(s^t,s_1\right)\right)}{P(s^t,s_1)} + \pi\left(s_2|s^t\right)\frac{u_C\left(R^1\left(s^t,s_2\right)\right)}{P(s^t,s_2)} + \\ ... + \pi\left(s_n|s^t\right)\frac{u_C\left(R^1\left(s^t,s_n\right)\right)}{P(s^t,s_n)} \end{array}\right] \tag{A.1}$$

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta^2 R^2\left(s^t\right)\left[\begin{array}{c} \pi\left(s_1|s^t\right)\frac{u_C\left(R^1\left(s^t,s_1\right)\right)}{\beta R^1(s^t,s_1)P(s^t,s_1)} + \pi\left(s_2|s^t\right)\frac{u_C\left(R^1\left(s^t,s_2\right)\right)}{\beta R^1(s^t,s_2)P(s^t,s_2)} + \\ ... + \pi\left(s_n|s^t\right)\frac{u_C\left(R^1\left(s^t,s_n\right)\right)}{\beta R^1(s^t,s_n)P(s^t,s_n)} \end{array}\right] \tag{A.2}$$

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta^3 R^3\left(s^t\right)\left[\begin{array}{c} \pi\left(s_1|s^t\right)\frac{u_C\left(R^1\left(s^t,s_1\right)\right)}{\beta^2 R^2(s^t,s_1)P(s^t,s_1)} + \pi\left(s_2|s^t\right)\frac{u_C\left(R^1\left(s^t,s_2\right)\right)}{\beta^2 R^2(s^t,s_2)P(s^t,s_2)} + \\ ... + \pi\left(s_n|s^t\right)\frac{u_C\left(R^1\left(s^t,s_n\right)\right)}{\beta^2 R^2(s^t,s_n)P(s^t,s_n)} \end{array}\right] \tag{A.3}$$

...

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta^n R^n\left(s^t\right)\left[\begin{array}{c} \pi\left(s_1|s^t\right)\frac{u_C\left(R^1\left(s^t,s_1\right)\right)}{\beta^{n-1} R^{n-1}(s^t,s_1)P(s^t,s_1)} + \pi\left(s_2|s^t\right)\frac{u_C\left(R^1\left(s^t,s_2\right)\right)}{\beta^{n} R^{n-1}(s^t,s_2)P(s^t,s_2)} + \\ ... + \pi\left(s_n|s^t\right)\frac{u_C\left(R^1\left(s^t,s_n\right)\right)}{\beta^{n-1} R^{n-1}(s^t,s_n)P(s^t,s_n)} \end{array}\right] \tag{A.4}$$

If $R^j\left(s^t\right)$, $j = 1, ..., n$ are set exogenously, for a given price level $P\left(s_t\right)$ these are n equations in n unknowns, $P\left(s^t, s_j\right)$, $j = 1, ..., n$. As long as there is enough variability in $R^j\left(s^t, s_j\right)$, $j = 1, ..., n$, that guarantees that the matrix of coefficients is invertible, there is a unique solution of the system of equations. For a given initial price level $P\left(s_0\right)$, there is a unique solution for the allocations and prices.

A.2. Term premia

The conditions (A.1) through (A.4) can be written as

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta R^1\left(s^t\right) E_t\left[\frac{u_C\left(R^1\left(s^{t+1}\right)\right)}{P\left(s^{t+1}\right)}\right] \tag{A.5}$$

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta R^j\left(s^t\right) E_t\left[\frac{u_C\left(R^1\left(s^{t+1}\right)\right)}{R^{j-1}\left(s^{t+1}\right)P\left(s^{t+1}\right)}\right], j = 2, ..., n \tag{A.6}$$

The conditions for $j = 2, ..., n$ can be rewritten as

$$\frac{u_C(R^1(s^t))}{P(s^t)} = \beta R^j(s^t)\left\{E_t\left[\frac{1}{R^{j-1}(s^{t+1})}\right]E_t\left[\frac{u_C(R^1(s^{t+1}))}{P(s^{t+1})}\right] + Cov_t\left(\frac{1}{R^{j-1}(s^{t+1})}, \frac{u_C(R^1(s^{t+1}))}{P(s^{t+1})}\right)\right\} \tag{A.7}$$

Using (A.5), to substitute for $\frac{u_C(R^1(s^t))}{P(s^t)}$ in (A.6), we obtain

$$\frac{R^1(s^t)}{R^j(s^t)} = E_t\left[\frac{1}{R^{j-1}(s^{t+1})}\right] + \frac{Cov_t\left(\frac{1}{R^{j-1}(s^{t+1})}, \frac{u_C(R^1(s^{t+1}))}{P(s^{t+1})}\right)}{E_t\left[\frac{u_C(R^1(s^{t+1}))}{P(s^{t+1})}\right]}, \; j = 2, ..., n. \tag{A.8}$$

A.3. Proposition 3.1

Let $R^j(s^t)$, $j = 1, ..., n$ be set exogenously. At any $t \geq 0$, for each state s^t, for a given $P(s^t)$, $C(s^t)$ and $L(s^t)$, there are n equations (A.1) through (A.4), n resource constraints, (2.11), and one intratemporal condition (3.3) to determine n consumption levels $C(s^{t+1})$, n levels of leisure $L(s^{t+1})$, and one price level $P(s^{t+1})$. As long as there is enough variability in $R^j(s^t, s_j)$, $j = 1, ..., n$, that guarantees that the matrix of coefficients is invertible, there is a unique solution of the system of equations. For $t = 0$, there is only one constraint, the resource constraint to determine $C(s_0)$ and $L(s_0)$. $P(s_0)$ is historically given. For a given initial $C(s_0)$, there is a unique solution for the allocations and prices.■