Debt Financing*

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January 2013

Abstract

We show that the predictions on the effects of financing the debt with taxes or inflation change when the reaction of households on their demands for money is taken into account. In the model, the households change the optimal interval between bond trades. In standard cash-in-advance models, the interval between trades is fixed, which implies an inelastic demand for money in the long run. With optimal trading intervals, the demand for money is elastic and has a better fit to the data. We find that the decrease in consumption is higher with optimal time intervals when an increase in government purchases is financed with inflation. According to the model, financing a 10% increase in government purchases with inflation implies a decrease in consumption of 3.4% with fixed intervals, but a decrease in consumption of 21% with endogenous intervals. The welfare losses are larger when the reaction of households is taken into account.

JEL Codes: E30, E40, E50.

Keywords: optimal debt structure, deficit, demand for money, cash management, taxes, inflation.

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1. Introduction

We show that taking into account the reaction of households on their demand for money substantially changes the predictions on financing the debt with taxes or inflation. We use a model in which households choose the use of financial services to change their demand for money. Higher inflation makes agents use more resources in financial services to decrease their demand for money. When the demand for money varies little, financing the debt with taxes or inflation implies similar effects. Taking into account the changes in the demand for money implies different predictions on consumption, output, and welfare.

Financing an increase in government purchases with inflation, for example, implies a small increase in output when the demand for money varies little. For this reason, the effects from financing the debt with taxes or inflation are similar. Considering the changes in the demand for money yields a larger increase in output because more resources must be used in financial services. Moreover, the decrease in welfare is larger when the demand for money is allowed to change.

The mechanism through which households change the demand for money is the frequency of bond sales. In a standard cash-in-advance, households sell interest-bearing bonds for money in every period to cover goods purchases for one period. This is the case, for example, in Cooley and Hansen (1989, 1991). However, the fact that agents make bond sales and use all money proceeds in one period implies small variation in velocity, which is contrary to what we observe in the data. Even with cash and credit goods and with different assumption about the expectations of shocks, velocity varies little, as shown in Hodrick, Kocherlakota, and Lucas (1991).

A solution for the small variation in velocity is to allow households to use the money from bond sales only after a larger time interval. This is the case of the models in Grossman and Weiss (1983) and Rotemberg (1984), in which households are able to use the money from bond sales only with a two-period interval. Alvarez, Atkeson, and Edmond (2009) increase the size of the time interval and show that a model with these characteristics is able to fit the short-run variation in velocity.

Having fixed intervals between bond sales, however, still implies small variation in velocity in the long run. With fixed intervals, an increase in inflation implies changes in the labor supply and in the use of credit and cash goods, but the real demand for money remains approximately unresponsive in the long run. Silva (2012) shows that allowing agents to change the interval between bond sales for money implies an elastic demand for money and a better fit to the long run data. Moreover, the estimates of the welfare cost of inflation increase from approximately zero to one percent when the interval of bond sales increase.

Here, we let households choose their interval between bond sales, as in Silva (2012). As a result, the households can change more easily their demand for money. We analyze the effects of different ways of financing the government debt. As governments usually obtain their tax revenues from distortionary taxation, we remove the possibility of lump sum taxation and introduce distortionary taxation.

Using a model that is better able to fit the long run data is especially important for us because our objective is to obtain predictions for the long run effects of financing the debt in different ways. We compare the predictions of the model with fixed and with endogenous time intervals between trades. We find that the model with endogenous time intervals implies different predictions for the effects of financing the debt with taxes and inflation.

For an increase of 10% in government purchases financed with taxes, we obtain that consumption decreases in similar magnitudes for fixed or endogenous time intervals, it decreases 4.3% with fixed intervals and 4.8% with endogenous intervals. On the other hand, the decrease in consumption is much higher with endogenous time intervals when the increase in government purchases is financed with inflation. According to the model, financing the increase in purchases with inflation implies a decrease in consumption of 3.4% with fixed intervals, but a decrease in consumption of 21% with endogenous intervals. The predicted welfare losses, therefore, are much larger when we consider the reaction of households to change their frequency of trades.

The reason for the difference in results with fixed or endogenous trading intervals is difference in the use of resources. The households in the model have to pay a cost to transform bonds into money, as in Baumol (1952) and Tobin (1956). We interpret this cost as a cost to obtain financial services.

In order to decrease the demand for money, the households have to divert more resources to financial services. An economy with higher inflation may have higher output because more of its output is used to cover financial services. For the 10% increase in government purchases above, output increases 0.58% with fixed trading intervals but it increases 9.7% with endogenous trading intervals. Output increases but this increase is used to increase financial services. An important service for the households because it allows them to decrease the real demand for money. But a service that does not increases welfare, as welfare is derived from consumption goods. Therefore, although output may increase, the welfare losses are always larger with endogenous trading intervals.

An analyst considering only fixed intervals would conclude that financing the debt with taxes or inflation would generate similar welfare losses. But taking into account that households change their demand for money, a fact observed in the data, leads to much higher predictions of welfare losses when the debt is financed with inflation.

2. The Model

We extend the general equilibrium Baumol-Tobin model in Silva (2012). Money must be used to purchase goods, only bonds receive interest payments, and there is a cost to transfer the money from bond sales to the goods market. As a result, households accumulate bonds for a certain time and exchange bonds for money infrequently. The infrequent sales of bonds for money occur as in the models of Grossman and Weiss (1983), Rotemberg (1984) and, more recently, Alvarez, Atkeson, and Edmond (2009). The difference is that the timing of the financial transfers is endogenous. We introduce a choice between taxes and inflation, made by the government, to finance a given value of government consumption.

Time is continuous and denoted by $t \in [0, \infty)$. At any moment there are markets for assets, for the consumption good and for labor. There are two assets, money and nominal bonds. The markets for assets and the market for the good are physically separated in this economy.

There is an unit mass of households that are infinitely lived and have preferences over consumption and leisure. Households have two financial accounts: a brokerage account in which they hold bonds and a bank account in which they hold money. We assume that readjustments in the brokerage account have a fixed cost. As only money can be used to buy goods, households need to maintain an inventory of money in their bank account large enough to pay for their flow of consumption expenditures until the next transfer of funds.

Firms are perfect competitors and hire labor to produce the good. There is a government that must finance government expenditures with income taxes or seigniorage.

2.1. Firms

At date t, the firms combine the labor supplied by each household i, $h_t(i)$ with $i \in [0, 1]$, to produce the good of date t. The production function is linear in the amount of labor used,

$$y_t = A \int_0^1 h_t(i) di,$$

where A is a technological parameter.

Each hour of labor costs firms $W_t(1+\tau)$, where W_t is the nominal wage received by the worker and τ is the labor tax. Per each unit of the good sold, firms receive the price P_t. As firms are perfect competitors, the real wage received by the households is

$$w_t \equiv \frac{W_t}{P_t} = \frac{A}{1+\tau}. \tag{1}$$

Since the real wage is time invariant, we drop the subscript t from w_t.

2.2. Households

There is a continuum of infinitely lived households with measure one. Each household sells hours of labor h_t, to the firms and receives labor income $W_t h_t$, which goes to the brokerage account. The funds deposited in the brokerage account cannot be used to buy goods but receive nominal interest r. The money in the bank account can be used to buy goods. The household chooses optimally when to transfer funds between accounts. The transfer of funds has a real fixed cost γ.

Household i decides consumption $c_t(i)$, labor supply $h_t(i)$, the dates when transfers to the bank are made $T_j(i)$, $j = 1, 2, ...$, money holdings in the bank account $M_t(i)$, and bond holdings in the brokerage account $B_t(i)$. Let $B_{T_j^-(i)}(i)$ and $M_{T_j^-(i)}(i)$ represent bonds and money holdings just before $t = T_j(i)$, and let $B_{T_j^+(i)}(i)$ and $M_{T_j^+(i)}(i)$ represent bonds and money holdings just after $t = T_j(i)$. Formally, $B_{T_j^-(i)}(i) \equiv \lim_{t\to T_j(i), t<T_j(i)} B_t(i)$ and $B^+_{T_j(i)}(i) \equiv \lim_{t\to T_j(i), t>T_j(i)} B_t(i)$. The definitions of $M_{T_j^-(i)}(i)$ and $M_{T_j^-(i)}(i)$ are similar.

Household i, has an initial endowment of wealth $\mathbb{W}_0(i)$ which is divided exclusively between money $M_0(i)$ in the checking account and $B_0(i)$ in the brokerage account, i.e. $\mathbb{W}_0(i) = M_0(i) + B_0(i)$. Define an holding period as the interval between any two consecutive transfer times, i.e. $[T_j(i), T_{j+1}(i))$, for $j = 1, 2, ...$ The first time household i adjusts its portfolio of bonds is $T_1(i)$ and its first holding period is

$[T_1(i), T_2(i)$.

As we concentrate on the steady state equilibrium, we assume without loss of generality that the distribution of the initial endowments of wealth among the households is such that the fraction of households that choose to readjust their portfolio of bonds is the same at any moment and that the duration of the holding periods N is the same across households.

The wealth level of household i, $\mathbb{W}_s(i)$ at date s is given by the sum of its assets holdings, at the last date the household readjusted its portfolio, plus the accumulated labor and interest income minus the accumulated consumption expenditure. Let $T_0(i) \equiv 0$. Formally, $\mathbb{W}_s(i)$ for $s \in (T_j(i), T_{j+1}(i)]$, $j = 0, 1, 2, ...$, is given by the expression

$$\begin{aligned} \mathbb{W}_s(i) \equiv\ & M^+_{T_j(i)}(i) + e^{(s-T_j(i))r} B^+_{T_j(i)}(i) - \int_{T_j(i)}^{s} P_t c_t(i)\, dt \\ & + \int_{T_j(i)}^{s} e^{(t-T_j(i))r} P_t w_t h_t(i)\, dt, \end{aligned} \tag{2}$$

At each date $T_j(i)$, $j = 1, 2, ...$, household i readjusts its portfolio. The restriction it faces is that the portfolio chosen plus the real cost of readjusting must be smaller or equal to the current wealth,

$$M^+_{T_j(i)}(i) + B^+_{T_j(i)}(i) + P_{T_j(i)}\gamma \leq \mathbb{W}_{T_j(i)}(i), \text{ for } j = 1, 2, ... \tag{3}$$

Additionally the household faces a cash in advance constraint,

$$\int_{T_j(i)}^{s} P_t c_t(i)\, dt \leq M^+_{T_j(i)}(i), \text{ for } s \in [T_j(i), T_{j+1}(i)), \text{ for } j = 0, 1, 2, ... \tag{4}$$

Using (2) and (3), we can write the budget constraint of household i as

$$\begin{aligned} M^+_{T_{j+1}(i)}(i) + B^+_{T_{j+1}(i)}(i) + P_{T_{j+1}(i)}\gamma \leq M^+_{T_j(i)}(i) + B^+_{T_j(i)} e^{(T_{j+1}(i)-T_j(i))r} \\ - \int_{T_j(i)}^{T_{j+1}(i)} P_t c_t(i)\, dt + \int_{T_j(i)}^{T_{j+1}(i)} P_t e^{(T_{j+1}(i)-t)r} w h_t(i)\, dt, \end{aligned}$$

for the holding period j, $j = 0, 1, 2, ...$

Let $Z_{T_{j+1}(i)} \equiv e^{-(T_{j+1}(i)-T_j(i))r}$ denote the price of a bond at $T_j(i)$ that pays 1 dollar at $T_{j+1}(i)$. Define $Q_{T_{j+1}(i)} \equiv Z_{T_1(i)} ... Z_{T_{j+1}(i)} = e^{-T_{j+1}r}$ as the price at 0 of a bond that pays 1 dollar at $T_{j+1}(i)$. If we multiply by $Q_{T_{j+1}(i)}$ the budget constraint of holding periods $j = 0, 1, 2, ..., k$ and add up all of them, we obtain

$$\begin{aligned} & \sum_{j=1}^{k} Q_{T_j(i)} \left(1 - Z_{T_{j+1}(i)}\right) M^+_{T_j(i)}(i) + Q_{T_{k+1}(i)} (B^+_{T_{k+1}(i)}(i) + M^+_{T_{k+1}(i)}(i)) \\ \leq & -\sum_{j=0}^{k} Q_{T_{j+1}(i)} \left(\int_{T_j(i)}^{T_{j+1}(i)} P_t c_t(i)\, dt + P_{T_{j+1}(i)}\gamma \right) \\ & + \sum_{j=0}^{k} Q_{T_{j+1}(i)} \left(\int_{T_j(i)}^{T_{j+1}(i)} P_t e^{(T_{j+1}(i)-t)r} w_t h_t(i)\, dt \right) + B^+_0(i) + Q_{T_1(i)} M^+_0(i). \end{aligned}$$

Using the definition of $Q_{T_j(i)}$ and the fact that $\lim_{T_k(i)\to\infty} Q_{T_k(i)}(B^+_{T_k(i)}(i) + M^+_{T_k(i)}(i)) = 0$ at the optimum, we get

$$\begin{aligned} & \sum_{j=1}^{\infty} Q_{T_j} \left[M^+_{T_j}(i) + P_{T_j}\gamma \right] + V_0(i) \\ \leq & \sum_{j=0}^{k} \int_{T_j(i)}^{T_{j+1}(i)} P_t e^{-rt} w_t h_t(i)\, dt + B^+_0(i) + Q_{T_1(i)} M^+_0(i). \end{aligned} \tag{5}$$

where $V_0(i) = Q_{T_1} \int_0^{T_1(i)} P_t c_t dt$. Expression (5) is the intertemporal budget constraint of household i.

Household i has an intertemporal utility function

$$\sum_{j=0}^{\infty} \int_{T_j(i)}^{T_{j+1}(i)} e^{-\rho t} u\left(c_t\left(i\right), h_t\left(i\right)\right) dt. \tag{6}$$

We take the momentary utility function to be a GHH utility function $u\left(c_t\left(i\right), h_t\left(i\right)\right) = \frac{1}{1-1/\eta}\left(c_t\left(i\right) - \frac{\left(h_t(i)\right)^{1+\chi}}{1+\chi}\right)^{1-1/\eta}$, from Greenwood, Hercowitz, and Huffman (1988). The household problem is to choose the vector $\left\{c_t\left(i\right), h_t\left(i\right), M_{T_j^+(i)}\left(i\right), T_j\left(i\right)\right\}$ that maximizes the intertemporal utility function (6) subject to intertemporal budget constraint (5) and the cash in advance constraint that we rewrite here as

$$\int_{T_j(i)}^{T_{j+1}(i)} P_t c_t\left(i\right) dt \leq M_{T_j(i)}^{+}\left(i\right). \tag{7}$$

Let λ be the lagrange multiplier of (5) and $\mu_{T_j(i)}$ the lagrange multiplier of (7). Let $t \in \left[T_j\left(i\right), T_{j+1}\left(i\right)\right)$ for all $j = 1, 2, ...$ The first order conditions with respect to $c_t\left(i\right)$, $h_t\left(i\right)$, $M_{T_j(i)}^{+}\left(i\right)$, and $T_j\left(i\right)$ are

$$e^{-\rho t}\left(c_t\left(i\right) - \frac{\left(h_t\left(i\right)\right)^{1+\chi}}{1+\chi}\right)^{-1/\eta} = \mu_{T_j(i)} P_t, \tag{8}$$

$$e^{-\rho t}\left(c_t\left(i\right) - \frac{\left(h_t\left(i\right)\right)^{1+\chi}}{1+\chi}\right)^{-1/\eta}\left(h_t\left(i\right)\right)^{\chi} = \lambda e^{-tr} P_t w_t,$$

$$\lambda Q_{T_j(i)} = \mu_{T_j(i)}, \tag{9}$$

$$
\begin{aligned}
&e^{-\rho T_j} u\left(c^-_{T_j(i)}, l^-_{T_j(i)}\right) - e^{-\rho T_j} u\left(c^+_{T_j(i)}, l^+_{T_j(i)}\right) + \lambda[-\dot{Q}_{T_j(i)} M^+_{T_j}(i) - Q_{T_j} \dot{M}^+_{T_j}(i) \\
&- \gamma\left(P_{T_j}\dot{Q}_{T_j} + Q_{T_j}\dot{P}_{T_j}\right) + P_{T_j(i)} e^{-rT_j(i)} w_{T_j(i)} h^-_{T_j(i)}(i) - P_{T_j(i)} e^{-rT_j(i)} w_{T_j(i)} h^+_{T_j(i)}(i)] \\
&+ \mu_{T_j}\left(\dot{M}^+_{T_j}(i) + P_{T_j} c^+_{T_j}\right) + \mu_{T_{j-1}}\left(-P_{T_j} c^-_{T_j}\right) \\
&= 0. \qquad (10)
\end{aligned}
$$

Conditions, (8)-(9), imply

$$
(h_t(i))^{\chi} = \frac{\lambda Q_t P_t w_t}{\lambda Q_{T_j(i)} P_t} = w e^{-r(t-T_j(i))} \equiv w^*_t(i). \qquad (11)
$$

This equation equates the marginal rate of substitution between leisure and consumption to the adjusted real wage, w^*_t. Since preferences are GHH, the supply of labor is only a function of the adjusted real wage. The adjusted real wage is a function of the real wage, the nominal interest rate and the distance to the previous transfer time. Even if the size of the holding period $T_{j+1}(i) - T_j(i)$ is the same for all households, the labor supply decreases within the holding period. This is so because the relevant real wage, $w^*_t(i)$, decreases within the holding period at a constant rate: $w^*_n = we^{-rn}$ for $n = [0, N]$.

2.3. Government

The government is continuously in the asset markets exchanging bonds for money. However, to derive the intertemporal government budget constraint it is convenient to assume a fictitious discrete timing economy, with intervals of dimension δ, which later we make arbitrarily small.

The government issues non state-contingent debt B_t and money M_t, makes consumption expenditures g, and taxes labor income at rate τ. At any moment t, total public debt D_t can take two forms. The government can choose to divide it between

M_t and B_t,

$$M_t + B_t = D_t.$$

The financial responsibilities of the government at time $t+\delta$ are equal to $M_t + B_t e^{\delta r} - P_t g + \tau P_t w h_t$. Thus, the time $t+\delta$ budget constraint of the government can be written as

$$M_{t+\delta} + B_{t+\delta} \leq M_t + B_t e^{\delta r} - P_t g + \tau P_t w h_t.$$

Similarly, for the $t + 2\delta$ budget constraint of the government,

$$M_{t+2\delta} + B_{t+2\delta} = M_{t+\delta} + B_{t+\delta} e^{\delta r} - P_{t+\delta} g + \tau P_{t+\delta} w h_{t+\delta}.$$

Multiplying the $t + \delta$ budget constraint by $e^{-\delta r}$, the time $t + 2\delta$ budget constraint $e^{-2\delta r}$, etc..., and adding up all of them, we get

$$\begin{aligned} & \sum_{s=0}^{k} \left(e^{-(s+1)\delta r} - e^{-(s+1)\delta r}\right) M_{t+(s+1)\delta} + e^{-k\delta r} \left(M_{t+k\delta} + B_{t+k\delta}\right) \\ = & \sum_{s=0}^{k} e^{-(s+1)\delta r} P_{t+s\delta} \left(\tau w h_{t+s\delta} - g\right) + B_t + e^{-\delta r} M_t. \end{aligned}$$

Since $e^{-k\delta r} \left(M_{t+k\delta} + B_{t+k\delta}\right) \longrightarrow 0$ as $k \longrightarrow \infty$, then

$$\sum_{s=0}^{\infty} \left(e^{-(s+1)\delta r} - e^{-(s+2)\delta r}\right) M_{t+(s+1)\delta} = \sum_{s=0}^{\infty} e^{-(s+1)\delta r} P_{t+s\delta} \left(w \tau h_{t+s\delta} - g\right) + B_t + e^{-\delta r} M_t.$$

As $\delta \longrightarrow 0$,

$$\begin{aligned} B_t + M_t \;\; = \;\; & r \int_t^{\infty} e^{-r(s-t)} M_s ds + \int_t^{\infty} e^{-r(s-t)} \tau P_s w h_s ds \\ & - \int_t^{\infty} e^{-r(s-t)} P_s g ds. \end{aligned}$$

Dividing both sides of the equation by P_t, we get the intertemporal budget constraint of the government,

$$b_0 + m_0 = r \int_0^{\infty} e^{(\pi - r)s} m_s ds + \int_0^{\infty} e^{(\pi - r)s} \tau w h_s ds - \int_0^{\infty} e^{(\pi - r)s} g ds. \tag{12}$$

2.4. Market Clearing Conditions

In equilibrium, all markets clear. Labor market clearing was already assumed to save on notation. The money demand is equal to the supply of money:

$$\int_0^1 M_t(i)\, di = M_t. \tag{13}$$

The demand for bonds by each household is equal to the total supply:

$$\int_0^1 B_t(i)\, di = B_t. \tag{14}$$

The clearing of the goods markets implies that aggregate private consumption plus public consumption plus financial services equals production:

$$g + \int_0^1 c_t(i) di + \frac{\gamma}{N} = y_t. \tag{15}$$

3. Solving for the Equilibrium

A competitive equilibrium is a sequence of policies, allocations and prices such that: (i) the private agents (firms and households) solve their problems given the sequences of policies and prices, (ii) the budget constraints of the government are satisfied and (iii) markets clear

We are interested in studying the steady state equilibria of this economy, as such we assume that the initial distribution of bonds and money among the households is such that the economy is in the steady state. The equilibrium steady state has the properties that all holding periods, have the same duration, N, and that all households behave similarly during their holding periods. Thus, all households readjust their portfolio in the same way, being equal the fraction of households that readjust their portfolio at any moment in this interval. Thus, household $i \in [0,1]$, which initially adjusts the portfolio at date $n(i) \in [0,N)$, will also readjust the portfolio at dates $n(i) + jN$ for $j = 1, 2, ...$

3.1. Consumption

We start by computing the consumption of household i. First we compute the equilibrium value of an artificial variable, $\mathfrak{c}_t$, given the value of this variable we get the equilibrium values of consumption $c_t(i)$ and of hours $h_t(i)$. From (8), (9) and (11) we get

$$\mathfrak{c}_t(i) \equiv \left(c_t(i) - \frac{\left(w_t^*(i)\right)^{\frac{1+\chi}{\chi}}}{1+\chi} \right) = \frac{e^{-\eta r\left(t - T_{j(i)}\right)}}{[\lambda P_0]^{\eta}}, \tag{16}$$

where we use the fact that the nominal interest rate, r, in the steady state is

$$r = \rho + \pi.$$

It is clear from (16) that $\mathfrak{c}_t$ decreases in the interval $t \in [T_j(i), T_{j+1}(i))$ at the rate ηr. Let $\mathfrak{c}_0$ be the value of $\mathfrak{c}_t$ at the beginning of the holding period, then

$$\mathfrak{c}_t(i) = \mathfrak{c}_0 e^{-\eta r\left(t - T_{j(i)}\right)} \text{ for } t \in [T_j(i), T_{j+1}(i)). \tag{17}$$

The equilibrium condition in the good market can be rewritten as

$$g + \int_0^1 \mathfrak{c}_t(i) di + \int_0^1 \frac{\left(w_t^*(i)\right)^{\frac{1+\chi}{\chi}}}{1+\chi} + \frac{\gamma}{N} = A \int_0^1 h(i)\, di.$$

We can solve for $\mathfrak{c}_0$ by replacing $\mathfrak{c}_t(i)$, given by (17), in the equation above,

$$\mathfrak{c}_0 = \frac{\eta r N}{\left(1 - e^{-\eta r N}\right)} \left(\frac{A w^{\frac{1}{\chi}}}{N} \frac{1 - e^{-\frac{rN}{\chi}}}{\frac{r}{\chi}} - g - \frac{w^{\frac{1+\chi}{\chi}}}{N\left(1+\chi\right)} \frac{1 - e^{-rN\frac{1+\chi}{\chi}}}{r\left(\frac{1+\chi}{\chi}\right)} - \frac{\gamma}{N} \right). \tag{18}$$

Given $\mathfrak{c}_0$ we obtain $\mathfrak{c}_t(i)$ from (17). Given w and $w_t^*(i)$ we obtain the equilibrium values for $c_t(i)$ from (16) and $h_t(i)$ from)11).

3.2. Holding Period

We obtain an expression for the size of the holding period N with the conditions (8), (9) and (10), and replacing μ_{T_j} with $\lambda Q_{T_j(i)}$. The steps for the derivation are in the Appendix. We obtain

$$\begin{aligned} & rN\mathfrak{c}_0 \left(\frac{e^{(\pi - \eta r)N} - 1}{\left(\pi - \eta r\right)N} - \frac{e^{-(\eta-1)rN} - 1}{-\left(\eta - 1\right)rN} \right) + \gamma\left(r - \pi\right) \\ & + rN\left(\frac{w^{\frac{1+\chi}{\chi}}}{1+\chi} \right) \left(\frac{e^{\left(\pi - r\frac{1+\chi}{\chi}\right)N} - 1}{\left(\pi - r\frac{1+\chi}{\chi}\right)N} - \chi \frac{1 - e^{-\frac{rN}{\chi}}}{rN} \right) = 0. \end{aligned} \tag{19}$$

Equation (19) yields the optimal size of the holding period N.

3.3. Money Demand

At any time t there will be households that are in their $j+1$ holding period while others are in their j holding period. For a household i that is in its $j+1$ holding period, the money demand is

$$M_t\left(i\right) = \int_t^{T_{j+2}(i)} P_0 e^{\pi z} c_z\left(i\right) dz$$

while the money demand for a household that is in its j holding period at time t is

$$M_t(i) = \int_t^{T_{j+1}(i)} P_0 e^{\pi z} c_z(i)\, dz.$$

Then, as shown in the appendix, the aggregate real money demand $\frac{M_t}{P_t} = \frac{1}{P_t}\int_0^1 M_t(i)\, di$ can be written as

$$\begin{aligned}\frac{M_t}{P_t} &= \frac{\mathfrak{c}_0}{N(\eta r - \pi)}\left[\frac{1-e^{-r\eta N}}{r\eta} - \frac{e^{-(\eta r-\pi)N}\left(1-e^{-\pi N}\right)}{\pi}\right] \\ &\quad + \frac{P_0 w^{\frac{1+\chi}{\chi}} e^{\pi t}}{N(1+\chi)\left(r\frac{1+\chi}{\chi}-\pi\right)}\left[\frac{1-e^{-r\frac{1+\chi}{\chi}N}}{r\frac{1+\chi}{\chi}} - \frac{e^{-r\frac{1+\chi}{\chi}N}\left(e^{\pi N}-1\right)}{\pi}\right]. \qquad (20)\end{aligned}$$

3.4. Equilibrium

Given $\{g, r\}$ the steady state equilibrium conditions for $\{w, \mathfrak{c}_0, N, \tau, m\}$ solve the system of equations (11), (12), (18), (19) and (20). The remaining equilibrium variables, $c_t(i)$ and $h_t(i)$ can be obtained from

$$\mathfrak{c}_t(i) = \mathfrak{c}_0 e^{-\eta r\left(t - T_{j(i)}\right)}, \text{ for } t \in [T_j(i), T_{j+1}(i))$$

$$h_t(i) = w e^{-\frac{r}{\chi}(t - T_j(i))}, \text{ for } t \in [T_j(i), T_{j+1}(i))$$

and

$$\left(c_t(i) - \frac{(h_t(i))^{1+\chi}}{1+\chi}\right) = \mathfrak{c}_t(i).$$

The following results are derived in the appendix. The first says that the equilibrium holding period increases if the transfer cost increases. And the second result says that the equilibrium holding period decreases if the nominal interest rate increases. Both of them are quite intuitive.

Result 1: $\frac{\partial N}{\partial \gamma} > 0$.

Result 2: $\frac{\partial N}{\partial r} < 0$.

4. Ramsey Problem

Here we study the Ramsey problem of this economy. Clearly, if lump-sum taxation was available setting the nominal interest rate equal to zero would be the first best. With distortionary taxation, the answer is less trivial. We investigate which is the best allocation associated with a steady state equilibrium. The Ramsey problem is

$$\max_{\{c_t(i),h(i),m,\tau,\pi\}} U^T \equiv \int_0^1 \frac{1}{1-1/\eta}\left(c_t\left(i\right) - \frac{\left(h_t\left(i\right)\right)^{1+\chi}}{1+\chi}\right)^{1-1/\eta} di$$

subject to five constraints: (11), (12), (18), (19) and (20). The objective function of this problem can be rewritten as

$$U^T = \left(\mathfrak{c}_0\right)^{1-1/\eta}\left[\frac{e^{(1-\eta)rN} - 1}{N\left(1-\eta\right)r}\right].$$

It is optimal for the government to make the initial total public debt equal to zero. So we assume, without loss of generality, that $b_0 + m_0 = 0$. Thus, in the steady state, public consumption is equal to the inflation tax plus the income tax,

$$rm + \tau w \int_0^1 h\left(i\right) di = g. \tag{21}$$

Result 3: The Friedman rule applies (Friedman 1969).

Proof (sketch): If $r = 0$ then all households equate their MRS between themselves and to the real wage w. The only remaining distortion is between the MRT and the MRS due to the tax, since $w = \frac{A}{(1+\tau)}$.

4.1. Welfare Cost of Inflation

Let $\bar{r}$ be the lower interest rate and ask by how much would consumers need to be compensated to be as well as after the interest rate increase to r. Let $U^T(\bar{r}, \overline{g}, \Delta) = (\mathfrak{c}_0)^{1-1/\eta}\left[\frac{e^{(1-\eta)rN}-1}{N(1-\eta)r}\right]$, where $\mathfrak{c}_0$ and N are the equilibrium values for the economy when the nominal interest rate and government expenditures are $\bar{r}$ and $\overline{g}$ respectively, and there is an exogenous transfer to each household of an extra flow of real income equal to Δ. The income compensation so that agents are indifferent between $\bar{r}$ and r is defined as $U^T(\bar{r}, \overline{g}, 0) = U^T(r, \overline{g}, \Delta)$. In the economy with variables $(r, \overline{g}, \Delta)$ the market clearing condition is

$$\overline{g} + \int_0^1 c_t(i)di + \frac{\gamma}{N} = A\int_0^1 h(i)\,di + \Delta.$$

The equation for consumption becomes

$$\mathfrak{c}_0 = \frac{\eta r N}{(1-e^{-\eta r N})}\left(\frac{Aw^{\frac{1}{\chi}}}{N}\frac{1-e^{-\frac{rN}{\chi}}}{\frac{r}{\chi}} + \Delta - \overline{g} - \frac{w^{\frac{1+\chi}{\chi}}}{N(1+\chi)}\frac{1-e^{-rN\frac{1+\chi}{\chi}}}{r\left(\frac{1+\chi}{\chi}\right)} - \frac{\gamma}{N}\right).$$

Of all the equilibrium equations, the equation above is the only one that changes. To the previous system of equilibrium allocations we add up one more variable, Δ, and one more equation

$$\frac{(\mathfrak{c}_0)^{1-1/\eta}}{1-1/\eta}\left[\frac{e^{(1-\eta)rN}-1}{N(1-\eta)r}\right] = U^T(r, \overline{g}, \Delta).$$

As we have as many equations as unknowns we can solve the system of equations.

4.2. Parameterization

The labor supply elasticity $\frac{1}{\chi}$, is set to 0.5. The highest value found in the literature is 1.6; lower values are validated by micro studies. The degree of risk aversion $1/\eta$, is set to 2. Usually, the estimates for η, the elasticity of intertemporal substitution, are above 0.1 and below 10. Government consumption is set equal to 20% of the output and the real interest rate to 3%. The cost of the transfer is set equal to the output produced during a quarter of a day. Using this parameterization we compute the welfare cost of increasing the inflation rate from 0% to 10%. The optimal value of N when inflation is 0% is about 3 months and 2 weeks, and it is 1 month and 3 weeks when inflation is 10%. The welfare cost of inflation, when N is endogenous is 0.38% of the output. The welfare cost of inflation, when N is fixed at 3 months and 2 weeks is 0.0049%.

5. Government Consumption Multiplier

We introduce physical capital and consider KPR preferences, from King, Plosser, and Rebelo (1988). The utility function is

$$u\left(c_t\left(i\right), h_t\left(i\right)\right) = \frac{\left[c_t\left(i\right)\left(1 - h_t\left(i\right)\right)^{\alpha}\right]^{1-1/\eta}}{1 - 1/\eta}.$$

We also assume that production requires physical capital in addition to labor. The production function is given by

$$Y_t = AK_t^{\theta} H_t^{1-\theta}.$$

We want to compute the effect over total consumption and output of an increase in government consumption financed by either i) an increase in the wage tax or ii) an increase in inflation.

5.1. Firms

Firms combine labor and capital to maximize profits. Profits are given by

$$P_t A K_t^{\theta} H_t^{1-\theta} - W_t H_t - r_t^k P_t K_t.$$

As firms are perfect competitors, the first order conditions are

$$W_t = (1-\theta) P_t A K_t^{\theta} H_t^{-\theta},$$

and

$$r_t^k = \theta A K_t^{\theta-1} H_t^{1-\theta}.$$

These equations can be rewritten as

$$\frac{W_t}{P_t Y_t} = (1-\theta) \frac{A K_t^{\theta} H_t^{-\theta}}{Y_t} = \frac{(1-\theta)}{H_t},$$

and

$$r_t^k K_t = \theta Y_t. \tag{22}$$

5.2. Households

Each household sells hours of labor h_t to the firms and receives labor income $W_t (1-\tau) h_t$, which goes to the bank account and rents capital K_t to the firms and receives dividends that go to brokerage account. The funds deposited in the brokerage account cannot be used to buy goods but receive nominal interest r_t. The money in the bank account can be used to buy goods. The household chooses optimally when to transfer funds between accounts. The transfer of funds has a real fixed cost γ.

The problem of the households is

$$\max \sum_{j=0}^{\infty} \int_{T_j}^{T_{j+1}} e^{-\rho t} \frac{\left[c_t(i)\left(1-h_t(i)\right)^{\alpha}\right]^{1-1/\eta}}{1-1/\eta} dt$$

subject to

$$\sum_{j=1}^{\infty} Q_{T_j(i)} \left[M^{+}_{T_j(i)}(i) + P_{T_j(i)}\gamma\right] \leq B^{+}_0(i) + P_0 k_0(i),$$

where

$$M^{+}_{T_j(i)}(i) = \int_{T_j(i)}^{T_{j+1}(i)} P_t \left[c_t(i) - w_t h_t(i)\right] dt,$$

for $w_t \equiv \frac{W_t}{P_t}(1-\tau)$.

Replacing the cash in advance constraint in the budget constraint we get

$$\sum_{j=1}^{\infty} Q_{T_j(i)} \left[\int_{T_j}^{T_{j+1}} P_t \left[c_t(i) - w_t h_t(i)\right] dt + P_{T_j}\gamma\right] \leq B^{+}_0(i) + P_0 k_0(i).$$

Among the first order conditions of this problem, we have

$$e^{-\rho t}\left[c_t\left(1-h_t\right)^{\alpha}\right]^{-1/\eta}\left(1-h_t\right)^{\alpha} - \lambda Q_{T_j} P_t = 0, \tag{23}$$

$$-e^{-\rho t}\left[c_t\left(1-h_t\right)^{\alpha}\right]^{-1/\eta} c_t \alpha \left(1-h_t\right)^{\alpha-1} + \lambda w(t) Q_{T_j} P_t = 0.$$

From these two conditions, we derive the intratemporal rate of substitution between leisure and consumption

$$\frac{(1-h_t)}{\alpha c_t} = \frac{1}{w_t}, \text{ for } t \in \left[T_j(i), T_{j+1}(i)\right).$$

As w_t is constant in the stead -state then leisure $l_t \equiv (1-h_t)$ and consumption grow at the same rate. Equation 23 implies $\frac{\dot{c}}{c} = -\eta r + \alpha(\eta-1)\frac{\dot{l}}{l}$ in the steady state. As

$\frac{\dot{c}}{c} = \frac{\dot{l}}{l} \equiv g_c$ then $\frac{\dot{c}}{c} = \frac{\dot{l}}{l} = -\frac{\eta r}{1-\alpha(\eta-1)}$, $\eta \neq 1+\frac{1}{\alpha}$. Therefore, $g_c > 0$ for $\eta > 1+\frac{1}{\alpha}$ and $g_c < 0$ for $\eta < 1+\frac{1}{\alpha}$.

The first order condition with respect to $T_j(i)$ implies

$$\begin{aligned} & \frac{rN}{\alpha}\left[\left(\frac{1}{\eta-1}\right)\left(\frac{e^{rN}-1}{rN}\right)+\frac{e^{\pi N}-1}{\pi N}\right]+\frac{\gamma}{w}(r-\pi) \\ = & \left[r\frac{e^{(\pi+g_h)N}-1}{(\pi+g_h)N}\left(1+\frac{1}{\alpha}\right)+\left(\frac{\frac{1}{\alpha}}{\eta-1}-1\right)(r+g_h)\left(\frac{e^{(r+g_h)N}-1}{(r+g_h)N}\right)\right]Nh_0. \end{aligned} \quad (24)$$

The first order conditions with respect to bonds and capital imply the standard non arbitrage condition

$$(r_t - \pi_t) = \left(r_t^k - \delta\right),$$

which says that the rate of return on bonds on the left hand side must be equal to the real return on physical capital on the right hand side. The households must be indifferent between investing in bonds or capital.

The money demand of an agent at time t that made $j+1$ transfers is $M_t(i) = \int_t^{T_{j+2}(i)} P_s[c_s(i) - w_s h_s(i)]\,ds$, while the money demand of an agent at time t that made j transfers is $M_t(i) = \int_t^{T_{j+1}(i)} P_s[c_s(i) - w_s h_s(i)]\,ds$. It can be shown using the households' intratemporal condition that the aggregate money demand at date t is

$$M_t = \frac{P_0(1+\alpha)c_0e^{\pi t}}{(g_c+\pi)}\left(e^{(g_c+\pi)N}\frac{1-e^{-\pi N}}{\pi N} - \frac{e^{g_cN}-1}{g_cN}\right) - \frac{P_0e^{\pi t}wN}{2}$$

and aggregate real balances M_t/P_t

$$\frac{M_t}{P_t} = \frac{(1+\alpha)c_0}{(g_c+\pi)}\left(e^{(g_c+\pi)N}\frac{1-e^{-\pi N}}{\pi N} - \frac{e^{g_cN}-1}{g_cN}\right) - \frac{wN}{2}. \quad (25)$$

5.3. Government

The budget constraint of the government is

$$r_t m_t + \tau (1 - \theta) Y_t = G_t.$$

5.4. Market clearing

The markets for the assets, the good and labor are in equilibrium. In particular in the good market, the total of the various demands, aggregate private consumption, government consumption, financial consumption plus aggregate investment is equal to total output

$$c_0 \left(\frac{e^{g_c N} - 1}{g_c N} \right) + G_t + \frac{\gamma}{N} + \dot{K}_t + \delta K_t = Y_t.$$

5.5. Solving for the Equilibrium

A competitive equilibrium is a sequence of policies, allocations, and prices such that (i) the firms and households solve their problems given the sequences of policies and prices, (ii) the budget constraints of the government are satisfied, and (iii) markets clear. As before we are interested in studying the steady state equilibria of this economy. Therefore, we assume that the initial distribution of bonds and money among the households is such that the economy is in the steady state.

We now obtain nine independent equilibrium static equations on the nine variables c_0, N, τ, m, h_0, w, Y, K, H whose solution is part of a steady state equilibrium. The equations are the production function

$$Y = AK^{\theta} H^{1-\theta},$$

equation 22, on the renting of physical capital by firms,

$$K = \frac{\theta Y}{\rho + \pi},$$

the equation for the demand of labor

$$w = \frac{(1-\tau)(1-\theta)Y}{H},$$

the supply of hours by households

$$H = 1 - (1-h_0)\frac{e^{g_l N} - 1}{g_l N},$$

the households intratemporal condition

$$(1-h_0) = \frac{\alpha c_0}{w},$$

the government budget constraint

$$rm + \tau(1-\theta)Y = g,$$

the good's market clearing condition

$$c_0\left(\frac{e^{g_c N} - 1}{g_c N}\right) = \left(Y - \frac{\gamma}{N} - G - \delta\frac{\theta Y}{\rho + \delta}\right),$$

the households' condition 24

$$\begin{aligned} & \frac{rN}{\alpha}\left[\left(\frac{1}{\eta - 1}\right)\left(\frac{e^{rN} - 1}{rN}\right) + \frac{e^{\pi N} - 1}{\pi N}\right] + \frac{\gamma}{w}(r - \pi) \\ = & \left[r\frac{e^{(\pi + g_h)N} - 1}{(\pi + g_h)N}\left(1 + \frac{1}{\alpha}\right) + \left(\frac{\frac{1}{\alpha}}{\eta - 1} - 1\right)(r + g_h)\left(\frac{e^{(r+g_h)N} - 1}{(r + g_h)N}\right)\right] N h_0, \end{aligned}$$

and the money demand

$$m = \frac{(1+\alpha)\, c_0}{(g_c+\pi)} \left(e^{(g_c+\pi)N} \frac{e^{-\pi N}-1}{-\pi N} - \frac{e^{g_c N}-1}{g_c N} \right) - \frac{wN}{2}.$$

5.6. An Increase in Government Consumption

We set standard values for the parameters. $\alpha = 0.5$, $\eta = 2$, and $\theta = 0.3$. Depreciation of capital is set to 10%. Government consumption is set to 20% of output. The real interest rate is set to 3% and the inflation rate to 2%. The transfer cost γ is set equal to the output produced during half of a day. In equilibrium, $N = 3$ weeks for a nominal interest rate of 5% and a tax rate of 29%.

Consider first the case in which the increase in public consumption is financed by the income tax. When N is fixed at its initial optimal level, a 10% increase in government consumption leads to an increase of 0.05% in output, while private consumption drops 4.3%. When N is endogenous, the holding period and the real money holdings decrease. The 10% increase in government consumption leads to an increase of 0.25% in output. Private consumption drops 4.8%. Capital, labor and taxes increase.

Consider now that the increase in public consumption is financed by seigniorage. With N fixed at its initial optimal level, a 10% increase in government consumption leads to a 0.58% increase in output. Inflation increases from 2% to 42% and consumption decreases 3.4%. With N endogenous, a 10% increase in government consumption leads to a 9.7% increase in the output. The equilibrium inflation increases from 2% to 19% and consumption decreases 21%.

6. Conclusions

We take into account that households react to different fiscal policies by changing the demand for money. The demand for money decreases when inflation increases. But households need to divert resources to financial services in order to decrease the demand for money.

The households change their demand for money by increasing the frequency of bond trades. In contrast, standard cash-in-advance models assume that the frequency of trades is fixed. Letting the frequency of trades vary implies a more elastic demand for money and a better fit to the data.

Taking into account the changes in the demand for money imply different predictions about the effects of an increase in government purchases and about the effects of different forms of debt financing. The government consumption multiplier is larger when the timing of the transactions is endogenous. The government purchases multiplier is much larger if public consumption is financed with seigniorage.

Appendix

To be completed.

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