Investment Decisions in Finite-lived Monopolies*

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Abstract

This paper studies the value and optimal timing for investment in a finite-lived monopoly. We extend the literature on option games by considering the cases of random and certain-lived monopolies. When compared to the duopoly and monopoly market structures, these new settings produce significantly different results. A certain-lived monopoly induces investment sooner than the duopoly, if the initial firm in the market faces the risk of being preempted and, on the contrary, can deter investment more than in monopoly case if the leader role is pre-assigned. A random-lived monopoly induces entry somewhere between the duopoly and monopoly cases. A higher uncertainty deters investment in all cases.

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1 Introduction

Investment under uncertainty models, proposed by the real options literature, were initially developed under the assumption that the option to invest is proprietary (e.g.: McDonald and Siegel (1986), Dixit and Pindyck (1994), Trigeorgis (1996)) with firms acting as if they operate in a monopolistic market. Several authors have proposed models that relax this assumption, allowing for competition, most assuming a duopolistic market structure (e.g.: Grenadier (1996), Pawlina and Kort (2006), Shackleton et al. (2004), Smets (1993)).[3] This growing stream of literature, known as option games, merges real options theory and game theory.[4] The main issue addressed by this literature is the optimal timing of investment when a firm operates in a competitive market.

Most of these models assume that the number of potential competitors is known and fixed, with their behavior being endogenously determined, and the market structure is only determined endogenously by the competitive game. For instance, in the case of a duopoly with preemption, the market can evolve from a monopoly (when the leader enters the market) to a duopoly (when the follower enters the market), but the number of firms than can enter the market is always two. However there are many examples of markets whose structure can change exogenously, for example from a monopoly to a competition market. One such example is a Government-granted monopoly, protected from competition by a non-perpetual barrier (e.g.: patent or exclusive right). European incumbent telecom companies, and several other utilities, were protected from competition until the Government opened the market in the last decades of the 20^{th} century. In some cases, the number of players allowed in market is still regulated. This type of protection is not exclusively sponsored by Governments. Firms can also grant a monopoly to other firms, through special rights, like an exclusive right to produce or sell a product in a certain market. For example, Apple, Inc gave AT&T the exclusive right to sell the iPhone in the US in 2007. However, in February 2011, Apple allowed Verizon to also sell the iPhone, ending AT&T's monopolistic right. Another example is the pharmaceutical industry, where firms license their patented drugs to other companies, that can exclusively produce or market the drug, in a certain country or region, until they concede other companies the same license.

Under this setting, besides the usual sources of uncertainty, firms have to deal with the possibility of a change in the market structure. They operate in a finite-lived monopoly, but face the risk of demonopolization, that can occur as a random or a certain event. The incumbent

[3] Models with more than two players are scarce, and most assume collusive behavior of firms or entry sequencing exogenously determined. Bouis et al. (2009) is one notable exception, modeling an oligopoly with endogenous entry sequencing of n firms.

[4] For a review on the literature of competitive option games, please refer to Chevalier-Roignant and Trigeorgis (2011), and Azevedo and Paxson (2012).

firm must adapt her behavior to account for the risk of losing the monopoly rents, producing a significant impact on the value.

This paper extends the previous literature on real options under competition by considering that the market structure is not a steady state, not allowed to change. Here a monopolistic firm faces the threat of demonopolization, that changes the market structure to a duopoly market. This threat is treated both as a random and a certain event, and impacts the monopolistic firm even when idle, wanting to enter the market.

Notice that in the duopolistic leader-follower setting, the market structure remains unchanged as a two players game. As the investment opportunity value increases, the number of active players in the market changes from zero to two. For certain values, it is only optimal for a single firm to be active in the market (the leader), benefiting from monopolistic rents, while waiting for the second player (the follower) to enter the market. From the beginning, the market has two places available, that are occupied sequentially by the firms. Typically these models assume that the equilibrium, translated into the entry timing, is endogenously determined.

The competitive setting studied in this paper, where a monopolistic firm faces the threat of a market structure change, does not fit in the existing duopolistic models. In this paper the number of places in the market may change, as a result of an exogenous event. Before that event, the firm acts in a monopolistic single player market, and a second player can only enter the market after that event.

Some early real options literature modeled firms entry in the market as an exogenous event. In Trigeorgis (1991) as more firms randomly enter the market, arriving according to a Poisson process, the project value drops by a certain amount. Accordingly, the project value is modeled as a mixed jump diffusion process. Similarly, but for perpetual investment options, Dixit and Pindyck (1994) model competitors arrivals as exogenous Poisson events. Martzoukos (2002) extends the approach to capture multiple sources of exogenous events with random effects.

Unlike these models, in this paper the exogenous demonopolization event does not necessarily imply a competitor entry, but only changes the market structure, i.e. the number of firms allowed to enter the market. The entry in the market is an endogenous game, that depends on this exogenous event.

Often demonopolization is the result of a significant modification of the Government or regulator policy. Regulator policies in an uncertain environment have been studied using a real options approach by Brennan and Schwartz (1982), Dixit and Pindyck (1994), and Teisberg (1993, 1994). None of these papers considers the possibility of a regulator controlling the number of players in the market. Our paper introduces this possibility, analyzing the impact of changing a monopoly to a duopoly, and shows how this change affects the behavior and value of firms in an uncertain environment.

The models proposed in this paper address several issues related with firm decisions and regulators policies. From a firm perspective, they allow to determine the value of a firm, and its optimal entry timing, in a finite-lived monopoly. They also enable to compute the appropriate compensation that should be paid by the monopoly grant issuer (firm or regulator) for allowing the market structure to change to a competitive market or to define the monopoly period that induce investment at a given level of demand. Additionally, the models may help a regulator mitigate monopolistic abuses by defining the type of finite-monopoly that induces a desired outcome, in terms of investment timing, and how it is affected, among others, by uncertainty.

We study three different settings for a monopoly market structure that can change to a duopoly. In the first model this change is treated as an exogenous random event. Under this setting an idle or active monopolistic firm, faces the threat of competition, benefiting from a random-lived monopoly. In the second model the event of demonopolization occurs at a certain date after which the initial firm, if idle, can be preempted in the market by a second firm, or may have to share the market with the second player. The initial firm in the market benefits from a certain-lived monopoly, under the risk of preemption. The third model also considers the case of a certain-lived monopoly, but the role of the leader is pre-assigned.

We compare these three new settings with the monopoly and duopoly market structures. The entry timing differs significantly among them, for different monopoly protection periods, and can vary from the Marshallian trigger to a trigger higher than in the monopoly case. Increasing the protection period induces an early investment in a certain-lived monopoly under the risk of preemption, from the duopoly solution to the limit of a perpetual protection, which induces investment at the Marshallian trigger, that produces a zero NPV. By contrast, for the certain-lived monopoly when the leader does not face the risk of being preempted, entry occurs later than would occur in the case of a monopoly. A random-lived monopoly induces entry somewhere between the duopoly and monopoly cases. We also show that a higher uncertainty deters investment in all cases.

This paper unfolds as follows. Section 2 presents the model for a random-lived monopoly. The certain-lived monopoly is presented in the following two sections, with section 3 treating the case of preemption, and section 4 the case where the role of leader is pre-assigned. A comparative statics and some policy implications are discussed in section 5. Finally, section 6 concludes.

2 Random-lived monopoly

Let us assume a monopolistic firm facing the threat of being forced to share the market with a second player, upon a random exogenous event, that changes the market structure from a monopoly to a duopoly. In this context, the incumbent firm has no control about the time when the demonopolization occurs, meaning that the period of time during which she can benefit from monopolistic rents is random. The exogenous event can be determined, for instance, by a regulator, that has the power to define the number of players in the market. Notice that the demonopolization can occur when the incumbent is either active or idle.

2.1 The incumbent firm under the threat of demonopolization

In this section we derive the model to determine the value of an active firm, granted with a monopoly, facing the threat of demonopolization, hereafter referred also as incumbent firm.

The valuation follows the standard backwards procedure, starting with the decision process for a follower, assuming that an incumbent is already in place, and that the random event demonopolizing the market has already occurred.

Let x be the total cash flow for the whole market evolving randomly according to a standard geometric Brownian motion (gBm) as follows:

$$dx = \alpha x dt + \sigma x dz \tag{1}$$

where $x > 0$, α and σ correspond to the trend parameter (the drift) and to the instantaneous volatility, respectively. Additionally, $\alpha < r$ is the drift in the equivalent risk-neutral measure and r is the risk-free rate. Finally dz is an increment of the Wiener process.[5]

The cash flows that a given firm receives, after investing the amount K, depends on the market share. We denote $D(i)$ as the market share of the first active firm in the market, where i corresponds to the number of active firms. For a market with two firms, $i = \{1, 2\}$, and $D(2) < D(1)$, ensuring that the market share of a single firm active in the market is higher than the her market share when sharing the market with another firm. For the sake of simplicity, we exclude the case a second-mover advantage, assuming that $0.5 \leqslant D(2) < D(1)$, and that the incumbent captures all the potential market cash flows, $D(1) = 1$.

The demonopolization process ends the monopoly period, allowing a second firm to enter the market, changing its structure to a duopoly. The optimal behavior of the firms results in a leader-follower setting as in Smets (1993) and Dixit and Pindyck (1994). Dropping the entry barrier, does not imply necessarily the immediate entry of the follower. In fact, the second firm will behave optimally entering only at a given threshold level of x, denoted by x_F[6].

The value of firms after demonopolization

After demonopolization occurs, the second player optimal behavior is the same as in the standard leader-follower duopoly setting, acting as a follower, since the incumbent is already in the market. Upon demonopolization, the incumbent ceases to be a monopolist. Instead, she assumes the leader position, waiting for the follower entry, receiving all the market cash flow $xD(1) = x$.

[5] Alternatively the model can be derived using a demand function as in Smets (1993), and Dixit and Pindyck (1994).

[6] The value functions and the trigger of the follower are the standard monopolistic solutions since she the proprietary option to capture a fraction of the market ($xD(2)$).

After the follower entry, her cash flow drops to $xD(2)$. Accordingly, the value function for the incumbent leader, $L_D(x)$, must satisfy the following non-homogeneous ordinary differential equation (o.d.e.):

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 L_D(x)}{\partial x^2} + \alpha x \frac{\partial L_D(x)}{\partial x} - r L_D(x) + x = 0 \tag{2}$$

Standard procedures enables us to determine the values of the leader and the follower (Smets 1993, Dixit and Pindyck 1994).[7]

Proposition 1. *The value of the incumbent firm, acting as a leader in the market, $L_D(x)$, is:*

$$L_D(x) = \begin{cases} \dfrac{x}{r-\alpha} - \dfrac{\beta_1}{\beta_1 - 1} K \left(\dfrac{x}{x_F}\right)^{\beta_1} & \text{for } x < x_F \\ \\ \dfrac{xD(2)}{r-\alpha} & \text{for } x \geqslant x_F \end{cases} \tag{3}$$

The value of the follower, $F_D(x)$, is given by:

$$F_D(x) = \begin{cases} \dfrac{K}{\beta_1 - 1} \left(\dfrac{x}{x_F}\right)^{\beta_1} & \text{for } x < x_F \\ \\ \dfrac{x(1-D(2))}{r-\alpha} - K & \text{for } x \geqslant x_F \end{cases} \tag{4}$$

where

$$x_F = \frac{\beta_1}{\beta_1 - 1} \frac{r-\alpha}{1-D(2)} K \tag{5}$$

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{6}$$

The lower branch of Equations (3) and (4) is simply the present value of the perpetual cash flows shared by the two firms. In the upper branch, the first part is the present value of the monopolistic single firm cash flows, and the second part captures the expected present value of the lost cash flows resulting from the follower entry.

The value of the incumbent firm before demonopolization

Before the demonopolization occurs, the incumbent firm is a monopolist facing the risk of demonopolization. This event, that modifies the market structure, is due to a third-party decision (e.g., Government, regulator or licensor), and, from the firm perspective, occurs randomly and

[7] All proofs of the propositions can be found in the Appendix.

is out of the control of the firm. In other words, the demonopolization process is a random exogenous event. We assume that the demonopolization follows a Poisson process with intensity λ.

The active monopolist value function, $L_R(x)$, must satisfy the following non-homogeneous o.d.e.:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 L_R(x)}{\partial x^2} + \alpha x \frac{\partial L_R}{\partial x} - rL_R(x) + x + \lambda \left[L_D(x) - L_R(x)\right] = 0 \tag{7}$$

where the last term of the left-hand side of the equation reflects the expected loss in value, for a infinitesimal period of time, due to the occurrence of a non-anticipated demonopolization process.

The solution to this o.d.e., as shown in the appendix, is:

Proposition 2. *The value of the incumbent firm under the threat of demonopolization is given by:*

$$L_R(x) = L_D(x) + \begin{cases} b_1 x^{\eta_1} & \text{for } x < x_F \\ \\ b_4 x^{\eta_2} + \dfrac{x\,(1 - D(2))}{r - \alpha + \lambda} & \text{for } x \geqslant x_F \end{cases} \tag{8}$$

where:

$$\eta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} \tag{9}$$

$$\eta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} \tag{10}$$

$$b_1 = \left(\frac{\lambda}{r - \alpha + \lambda}\frac{\eta_2 - 1}{\eta_1 - \eta_2} + \frac{\beta_1 - \eta_2}{\eta_1 - \eta_2}\right)\frac{\beta_1}{\beta_1 - 1} K x_F^{-\eta_1} \tag{11}$$

$$b_4 = \left(\frac{\lambda}{r - \alpha + \lambda}\frac{\eta_1 - 1}{\eta_1 - \eta_2} + \frac{\beta_1 - \eta_1}{\eta_1 - \eta_2}\right)\frac{\beta_1}{\beta_1 - 1} K x_F^{-\eta_2} \tag{12}$$

Comparing with a firm already in a duopoly ($L_D(x)$), which corresponds to the situation after demonopolization, Equation (8) suggests that the firm under the threat of demonopolization is more valuable. In fact, since demonopolization is not certain, the incumbent may remain a monopolist for a longer a period of time.

Proposition 3. *An incumbent firm facing the threat of a demonopolization ($\lambda > 0$) is more valuable than a leader in a duopolistic market:*

$$L_R(x) \geqslant L_D(x) \tag{13}$$

Equation (8) shows the extra value coming from the possibility of remaining a monopolist beyond the trigger of the follower. This occurs because the follower can not enter the market, since she faces an entry barrier. The second term of the lower branch captures the cash flows seized by the incumbent until demonopolization occurs, if they remained forever above x_F, and the other terms in both branches correspond the value associated with the possibility of x crossing x_F, either from above or from below.

From proposition 2, two particular cases can be highlighted: when demonopolization is imminent ($\lambda = \infty$) and when it is impossible ($\lambda = 0$):

Proposition 4. *When $\lambda = \infty$, demonopolization is imminent, and the incumbent firm value converges to the leader value in a standard leader-follower duopoly:*

$$\lim_{\lambda \to \infty} L_R(x) = L_D(x) \tag{14}$$

Proposition 5. *When $\lambda = 0$ demonopolization is impossible and the incumbent value is simply the present value of the monopolistic cash flows:*

$$\lim_{\lambda \to 0} L_R(x) = \frac{x}{r - \alpha} \tag{15}$$

Besides the traditional market uncertainty related with the cash flows (σ), the incumbent firm faces an additional source of uncertainty related with the possibility of a change of the market structure (λ). The value of an active incumbent firm under the risk of a market structure change, allowing for the entry of another competitor, lies between the monopolistic value and the leader value in a duopoly.

2.2 The idle granted monopolist under the threat of demonopolization

In the previous section we valued an active incumbent firm facing the threat of having another competitor in the market. Consider now a firm granted with monopoly rights, waiting for the optimal time to enter the market. In absence of demonopolization risk, the firm holds a perpetual investment opportunity, with the well known solution (McDonald and Siegel 1986):

$$O_M(x) = \begin{cases} \dfrac{K}{\beta_1 - 1} \left(\dfrac{x}{x_R} \right)^{\beta_1} & \text{for } x < x_M \\ \\ \dfrac{x}{r - \alpha} - K & \text{for } x \geqslant x_M \end{cases} \tag{16}$$

where $O_M(x)$ is the value function for the perpetual monopolist, and

$$x_M = \frac{\beta_1}{\beta_1 - 1}(r - \alpha)K \tag{17}$$

represents the optimal trigger to invest.

If the idle granted monopolist faces the risk of a demonopolization, modeled as a Poisson process with intensity λ, her value, $O_R(x)$, differs from $O_M(x)$. During the continuation period (when it is not yet optimal to invest), $O_R(x)$ must satisfy the following o.d.e.:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 O_R(x)}{\partial x^2} + \alpha x \frac{\partial O_R(x)}{\partial x} - rO_R(x) + \lambda[\Gamma(x) - O_R(x)] = 0 \tag{18}$$

where:

$$\Gamma(x) = \begin{cases} F_D(x) & \text{for } x < x_D \\ \\ p\,[L_D(x) - K] + (1-p)F_D(x) & \text{for } x_D \leqslant x < x_R \end{cases} \tag{19}$$

The rationale for $\Gamma(x)$ is the following. Remember that the granted monopolist is idle, waiting for the optimal timing to invest. If the demonopolization occurs during this period, a second firm is allowed to enter the market, and the monopolistic investment opportunity becomes a leader-follower investment game. Assuming both firms are *ex ante* symmetric, for $x < x_D$, i.e., for an x lower that the optimal trigger for the first firm to enter the market (the leader), they both prefer to be the follower. In this region, the follower value function dominates the leader value function net of the investment cost ($F_D(x) > L_D(x) - K$). After x_D the leader position is more valuable ($F_D(x) < L_D(x) - K$), and both roles have the same value at $x = x_D$. The trigger for the leader, x_D, is determined numerically by solving the equation $F_D(x_D) = L(x_D) - K$. After that point, both firms prefer to be the leader, so they both decide to invest. However, only one of them effectively enters the market, achieving the leader position. The other firm acts optimally deferring the investment until x hits x_F, where x_F is the optimal trigger for the follower, given by Equation (5).

If demonopolization occurs in the interval where $x_D \leqslant x < x_R$, we assume that the granted monopolist has a probability p to enter the market as the leader, and $(1-p)$ to become the follower. If the firms are perfectly symmetric *ex ante*, both will have the same probability to become the leader ($p = 0.5$). However, we allow for different values of p, to cover, for example, the case where the granted monopolist, managing the option to invest, has some advantage over the potential competitor ($p > 0.5$).

Proposition 6. *The value of the option to invest of an idle granted monopolist under the threat* *of a demonopolization, exchanging K for the cash flows, is:*

$$O_R(x) = \begin{cases} a_1 x^{\eta_1} + F_D(x) & \text{for } x < x_D \\ a_3 x^{\eta_1} + a_4 x^{\eta_2} + p\left[L_D(x) - \dfrac{x}{r-\alpha+\lambda} - \dfrac{\lambda}{r+\lambda}K\right] & \\ \quad +(1-p)F_D(x) & \text{for } x_D \leqslant x < x_R \\ L_R(x) - K & \text{for } x \geqslant x_R \end{cases} \tag{20}$$

The four unknowns (the constants a_1, a_3, a_4, and the trigger x_R) are determined by solving numerically and simultaneously the four non-linear equations, that ensure $O_R(x)$ is continuous and differentiable along x:

$$a_1 x_D^{\eta_1} + F_D(x_D) = a_3 x_D^{\eta_1} + a_4 x_D^{\eta_2} + pH(x_D) + (1-p)F_D(x_D) \tag{21}$$

$$\eta_1 a_1 x_D^{\eta_1 - 1} + F'_D(x_D) = \eta_1 a_3 x_D^{\eta_1 - 1} + \eta_2 a_4 x_D^{\eta_2 - 1} + pH'(x_D) + (1-p)F'_D(x_D) \tag{22}$$

$$a_3 x_R^{\eta_1} + a_4 x_R^{\eta_2} + pH(x_R) + (1-p)F_D(x_R) = L_R(x_R) - K \tag{23}$$

$$\eta_1 a_3 x_R^{\eta_1 - 1} + \eta_2 a_4 x_R^{\eta_2 - 1} + pH'(x_R) + (1-p)F'_D(x_R) = L'_R(x_R) \tag{24}$$

where $H(x) = L_D(x) - \dfrac{x}{r-\alpha+\lambda} - \dfrac{\lambda}{r+\lambda}K$, $L'_R(x_R) = \dfrac{\partial L_R(x)}{\partial x}|_{x=x_R}$, $H'(x_j) = \dfrac{\partial H(x)}{\partial x}|_{x=x_j}$, $F'_D(x_j) = \dfrac{\partial F_D(x)}{\partial x}|_{x=x_j}$, and $j \in \{D, R\}$.

Upon investing K, at the trigger x_R, the granted monopolist becomes the incumbent active firm obtaining L_R (lower branch of Equation (20)). If $x < x_D$ (upper branch), the value of the idle granted monopolist is simply the value of the follower added to the value related with the possibility of x becoming greater that x_D. For intermediate values of x, we need to consider the chance of a future a demonopolization, changing the market structure to a duopoly. As explained before, the granted monopolist will be either the leader or the follower. If she becomes the leader, she will invest K obtaining $L_D(x)$. However, $L_D(x)$ is the value of a leader entering immediately the market. Given that this is only a probabilistic future event, the expression in square brackets measures the benefit of becoming the leader, net of the value of the deferred cash flows and investment cost. The first two terms capture the value of the possibility of x being lower than x_D and greater than x_R.

Analyzing the two limiting cases, when demonopolization is imminent ($\lambda = \infty$) and when it is impossible ($\lambda = 0$), from Proposition 6 we obtain:

Proposition 7. *When $\lambda = \infty$, demonopolization is imminent, the granted monopolist firm trigger and value converge to those of the standard leader-follower duopoly:*

$$\lim_{\lambda \to \infty} x_R = x_D \tag{25}$$

$$\lim_{\lambda \to \infty} O_R(x) = \begin{cases} F_D(x) & \textit{for } x < x_R = x_D \\ \\ L_D(x) - K & \textit{for } x \geqslant x_R = x_D \end{cases} \tag{26}$$

Proposition 8. *When $\lambda = 0$, demonopolization is impossible, the trigger and value of the granted monopolist are those of the monopolistic firm:*

$$\lim_{\lambda \to 0} x_R = x_M \tag{27}$$

$$\lim_{\lambda \to 0} O_R(x) = O_M(x) \tag{28}$$

These two propositions suggest that the investment trigger of the granted monopolist, in a random-lived monopoly, occurs somewhere between the duopoly and monopoly triggers.

3 Certain-lived monopoly under preemption

In the previous section the monopoly period protection was treated as a random variable. However, in some cases, that period has a certain finite duration, after which another player is allowed to enter the market. Typical examples of this setting are patents protecting a firm from competition during a defined period of time, and exclusive rights to sell a product or service in a given market (e.g.: the pharmaceutical industry).

Under this setting, let us assume that two firms are willing to invest. The first to enter the market, the leader, is granted with a monopoly for a certain finite period of time, starting at the investment date.[8] After the protection period, a second firm is allowed to enter the market. This firm, acting as a follower, invests when it becomes optimal, i.e. when the trigger to invest is achieved, but she is only allowed to do so after the monopoly period ends. Although the trigger can be reached before that, the follower is not allowed to enter the market, given that a barrier protects the incumbent from competition. The follower is, therefore, entitled with a forward start option to invest.

In order to obtain the value and investment trigger for the leader, we need to start from the optimal behavior of the second player, using a backward procedure.

3.1 The value of the forward start follower

Consider a firm with the option to enter a market protected from competition during T years. This is equivalent to a forward start call option, i.e. an option that can only be exercised after

[8] For the case of patented protected projects, this is equivalent to assume that the R&D stage is instantaneous and has no technical uncertainty. Alternatively we could consider the possibility of the monopolistic period having a fixed ending date, independent of the time of investment, but we would have to resort to numerical methods, departing from the quasi-analytical solutions presented in this paper.

T, as soon as the trigger x_F is achieved. Under the risk neutral expectation the value of the option is:

$$F_C(x) = e^{-rT} E\left[F_D(x(T))\right] \tag{29}$$

where $F_D(x(T))$ is the value of the follower's option to invest at time T. At that moment, the state variable x at T $(x(T))$ can either be below or above the trigger x_F given by Equation (5). For the latter case, she will invest K in exchange of the present value of future cash flows, which is similar to a European call option with maturity T exercised if $x(T) > x_F$. However, if the firm does not invest at time T, she still has the option to invest later on, which gives her an additional value.

Proposition 9. *The value of the forward start follower is given by:*

$$F_C(x) = \frac{x(1-D(2))}{r-\alpha} e^{-(r-\alpha)T} N\left(d_1(x)\right) - K e^{-rT} N\left(d_2(x)\right) + \frac{K}{\beta_1 - 1}\left(\frac{x}{x_F}\right)^{\beta_1} N\left(-d_3(x)\right) \tag{30}$$

where $N(.)$ is the cumulative normal integral and

$$d_1(x) = \frac{\ln\left(\frac{x}{x_F}\right) + \left(\alpha + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \tag{31}$$

$$d_2(x) = d_1(x) - \sigma\sqrt{T} \tag{32}$$

$$d_3(x) = d_1(x) + (\beta_1 - 1)\,\sigma\sqrt{T} \tag{33}$$

The first two terms of Equation (30) is the Black-Scholes formula for an option on a dividend paying asset with maturity T, exercised if $x(T) > x_F$, instead of $x(T) > K$ in the standard formula, which is reflected in the conditional probabilities $N(d(1,2))$. The last term captures the value of exercising the option to invest in a later stage (after T) if the trigger x_F is not reached at T $(x(T) < x_F)$.

As previously, the next two propositions show the value of the forward start follower for the two limiting cases of T.

Proposition 10. *When $T = \infty$, the monopoly protection period is perpetual, the follower will never be allowed to enter the market, and so, she becomes worthless:*

$$\lim_{T\to\infty} F_C(x) = 0 \tag{34}$$

Proposition 11. *When $T = 0$, there is no protection period, the follower behaves as in the standard duopoly case, and her value is:*

$$\lim_{T\to 0} F_C(x) = F_D(x) \tag{35}$$

3.2 The certain-lived monopolist

By investing K, the leader becomes a monopolist for a certain period of time T, after which the market changes to a duopoly, where a follower behaves optimally by entering the market when $x > x_F$. Comparing with the value of perpetual monopolist, the value of certain-lived monopolist is reduced by the possibility of the follower entering the market at T or afterwards.

Proposition 12. *The value of the leader with a certain monopolistic period is given by:*

$$\begin{aligned} L_C(x) = \; & \frac{x}{r-\alpha} - \frac{x\left(1-D(2)\right)}{r-\alpha} e^{-(r-\alpha)T} N\left(d_1(x)\right) \\ & - \frac{\beta_1}{\beta_1 - 1} K \left(\frac{x}{x_F}\right)^{\beta_1} N\left(-d_3(x)\right) \end{aligned} \tag{36}$$

where $N(.)$ is the cumulative normal integral and $d_1(x)$ and $d_3(x)$ are as defined before.

The first term of Equation (36) is the value of a perpetual monopolist. The lost value due to the possibility of having to share the market with a second firm is captured by the last two terms. The second term reflects the expected lost value if the follower enters the market at T (which occurs for $x(T) > x_F$) while the last one is the value lost accruing from the possibility of the follower entering at a later stage ($x(T) < x_F$).

The optimal timing for a firm to enter the market and benefit of the monopolistic rents as a leader, is reached as soon x hits the trigger, x_C. For that trigger the value of the forward start follower must be the same as the value of the leader net of the investment cost K. The trigger for the leader is determined numerically by solving $L_C(x) - K = F_C(x)$.

Analyzing the two extreme cases for the monopolistic period, it is possible to obtain the limiting values of certain-lived monopolist.

Proposition 13. *When $T = \infty$, the monopolistic period is perpetual, the certain-lived monopolist value converges to the value of the perpetual monopolist, and the trigger for investment is the Marshallian trigger:*

$$\lim_{T\to\infty} L_C(x) = \frac{x}{r-\alpha} \tag{37}$$

$$\lim_{T\to\infty} x_C = (r-\alpha)K = x_{NPV} \tag{38}$$

When a firm, upon investing, is entitled with a perpetual monopolistic period, the solution is the value of the perpetual cash flows, since the leader will never have to share the market with other firms. When two firms compete for the role of leader remaining monopolist forever, the optimal strategy is to enter immediately at the trigger corresponding to a zero net present value (x_{NPV}), otherwise she will be preempted by the other firm. Is such case the value of the follower is null, given that there is only one place in the market, and competition for that place destroys completely the value of the option to defer.

Proposition 14. *When $T = 0$, the certain-lived monopolist value and trigger are the standard leader-follower duopoly solutions:*

$$\lim_{T\to 0} L_C(x) = L_D(x) \tag{39}$$

$$\lim_{T\to 0} x_C = x_D \tag{40}$$

When the monopolistic period is reduced to zero, the market becomes immediately a duopoly and, therefore, the solution must be the same of the standard leader-follower duopoly, with the trigger for investment occurring when the value of leader net of the investment cost equals the value of the follower.

4 Certain-lived monopoly without preemption

In the previous section it was assumed that both firms contemplating investing in the market, disputed the roles of leader and follower. However, in some circumstances, these roles are exogenously pre-assigned. That is the case, for instance, when a firm is protected from competition by a patent or an exclusive right to market a product or service, that excludes the risk of being preempted in the market.[9] Under this setting, a pre-assigned leader has the valuable option to defer investment until it becomes optimal, investing at the myopic trigger of the monopolist (Chevalier-Roignant and Trigeorgis 2011, ch. 12).

Proposition 15. *The value of the pre-assigned leader in a duopoly is given by:*

$$O_{DA}(x) = \begin{cases} \dfrac{K}{\beta_1 - 1}\left(\dfrac{x}{x_{DA}}\right)^{\beta_1} - \dfrac{\beta_1}{\beta_1 - 1}K\left(\dfrac{x}{x_F}\right)^{\beta_1} & \textit{for } x < x_{DA} \\ L_D(x) - K & \textit{for } x \geqslant x_{DA} \end{cases} \tag{41}$$

where $x_{DA} = \dfrac{\beta_1}{\beta_1 - 1}(r - \alpha)K = x_M$ is the trigger for investment of the pre-assigned leader and corresponds to the myopic monopoly trigger.

When the pre-assigned leader benefits form a certain monopolistic protection period, we obtain the following value function and investment trigger:

Proposition 16. *The value of the pre-assigned leader with a certain monopolistic period is*

[9] We assume that firms are *ex-ante* symmetric, and, therefore, the only competitive advantage is due to these patents or exclusive rights. Extending the model to allow for asymmetric firms, for instance with a firm with a clear cost-advantage is another case where there is no risk of preemption.

given by:

$$O_{CA}(x) = \begin{cases} c_1 x^{\beta_1} & \text{for } x < x_{CA} \\ \\ L_C(x) - K & \text{for } x \geqslant x_{CA} \end{cases} \tag{42}$$

where x_{CA} is the solution to the following non-linear equation:

$$\frac{x_{CA}}{r-\alpha}\Big(1 - (1 - D(2))\, e^{-(r-\alpha)T} N\left(d_1(x_{CA})\right)\Big) - \frac{\beta_1}{\beta_1 - 1} K = 0 \tag{43}$$

and $c_1 = \left(L_C(x_{CA}) - K\right) x_{CA}^{-\beta_1}$.

In the limiting cases for the monopolistic protection period, the pre-assigned certain-lived monopolist invests at the same trigger as the pre-assigned leader in a duopoly.

Proposition 17. *When $T = \infty$, the monopolistic period is perpetual, and the pre-assigned certain-lived monopolist trigger converges to the trigger of the monopolist:*

$$\lim_{T\to\infty} x_{CA} = x_M \tag{44}$$

When the protection period is perpetual, the market will remain a monopoly forever, and therefore, the pre-assigned leader invests at the monopolistic trigger, that is also the trigger of the pre-assigned leader in a duopoly ($x_M = X_{DA}$).

Proposition 18. *When $T = 0$, the pre-assigned certain-lived monopolist trigger converges to the trigger of the pre-assigned leader in a duopoly:*

$$\lim_{T\to 0} x_{CA} = x_{DA} \tag{45}$$

When the pre-assigned certain-lived monopolist does not benefit of any protection, the market becomes a duopoly immediately, and the leader trigger converges, as expected, to the trigger of the pre-assigned leader in a duopoly.

The monopoly protection period is a barrier to the entry of a second firm in the market, giving the leader an additional advantage when compared to the duopoly case, where such protection does not exist, and the monopoly period is only determined endogenously by the behavior of both firms, ending at the follower trigger. This additional advantage deters the pre-assigned leader to invest sooner than wath would happen in the absence of any protection.

Proposition 19. *The pre-assigned leader in a certain-lived monopoly invests later than the pre-assigned leader in a duopoly:*

$$x_{CA} > x_{DA} (= x_M) \tag{46}$$

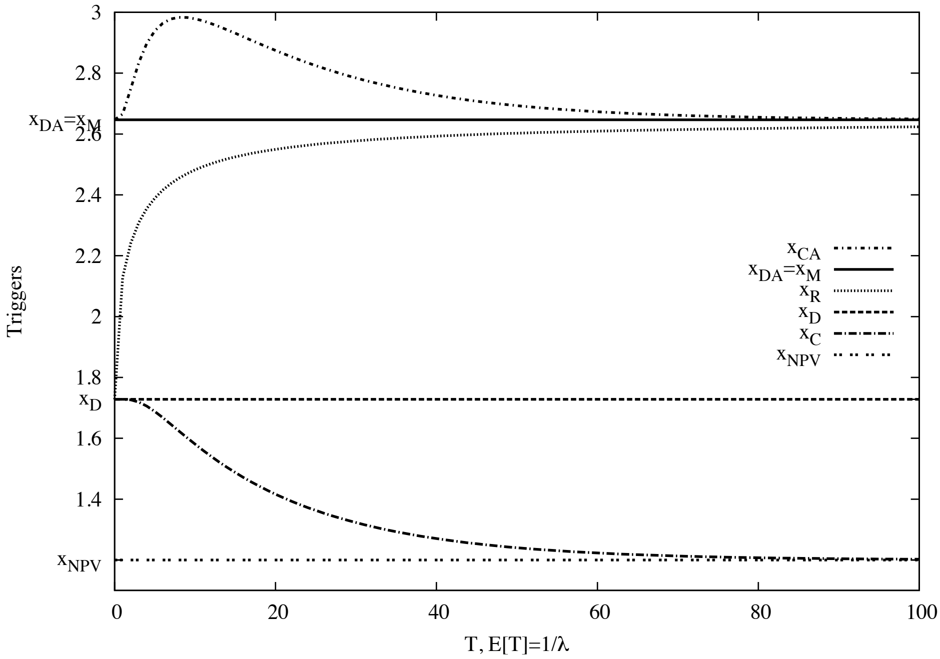

Figure 1: Investment triggers as a function T. $\sigma = 0.3$, $r = 0.05$, $\alpha = -0.01$, $K = 20$, $D(1) = 1$, $D(2) = 0.5$, $p = 0.5$.

5 The effect of monopoly duration and uncertainty

After presenting the models for the different finite-lived monopolistic settings, we perform, in this section, a comparative statics, focusing on the effects of the duration of the monopolistic protection period and its interaction with uncertainty, and analyze some implications for the firms behavior and regulator policies.

The limiting cases for the duration of the monopoly period (T) and the risk of demonopolization (λ) have been presented in the previous sections. Here we show how the triggers and value functions behave for different levels of these parameters. In order to compare their effects we need to define λ (the intensity rate of the Poisson process) in terms of time, using the expected arrival time, i.e $E[T] = 1/\lambda$.

Figure 1 illustrates the effect of the duration of the monopolistic period. Depending on the type of monopoly protection, the monopoly duration can hasten or deter investment (Figure 1). Investment occurs sooner for the certain-lived monopoly with preemption, and the trigger decreases with T, from the trigger in a duopoly (x_D) to the cash flows that produce a zero NPV (x_{NPV}). Increasing the protection period, makes investment for the leader more profitable, as the firm benefits from monopolistic rents for a longer period of time, reducing the threshold for investment. When two firms compete for a single place in a perpetual monopoly, the solution

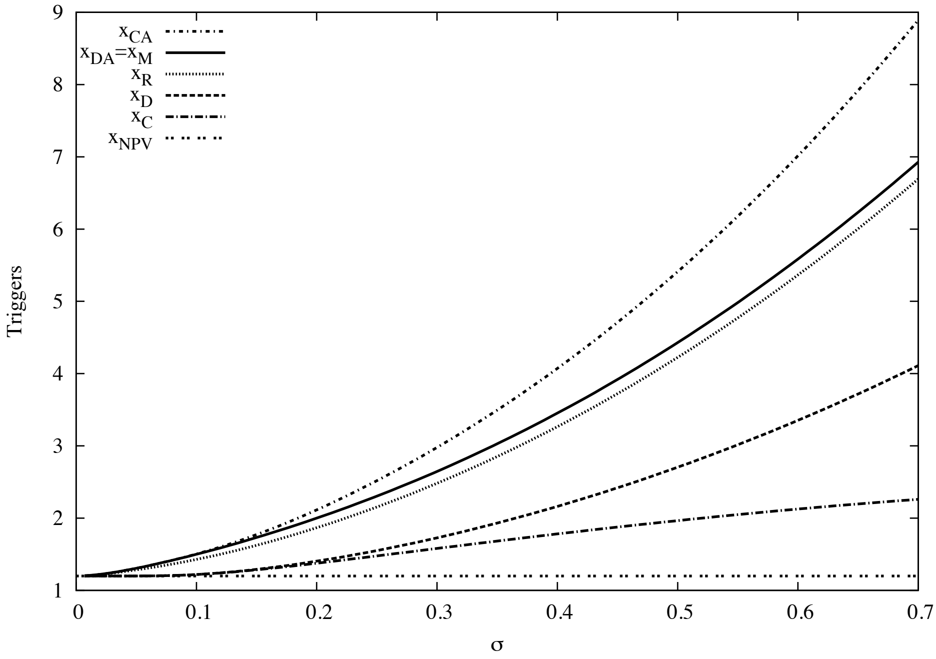

Figure 2: Investment triggers as a function σ. $r = 0.05$, $\alpha = -0.01$, $K = 20$, $D(1) = 1$, $D(2) = 0.5$, $T = 10$, $\lambda = 0.1$, $p = 0.5$.

is the perfect competitive solution, where investment occurs for a null NPV.

When the role of the leader is pre-assigned, we have precisely the opposite case, where investment is mostly deterred. For this setting, investment occurs always later than the duopoly with a pre-assigned leader solution, with the trigger initially increasing with T, starting from the duopoly trigger (x_{DA}), and afterwards begins decreasing with T, converging again to the duopoly solution. The effect of T on the behavior of the assigned leader is twofold: on the one hand a higher monopolistic period induces investment for lower levels of the state variable x, and on the other hand, by deterring investment, the leader also deters the follower entry, reducing, therefore, the penalty of delaying her own entry. These two opposing effects dominate each other for different levels of T, explaining the non-monotone behavior of x_{DA} in relation to T.

In a random-lived monopoly, optimal investment occurs later as λ decreases (i.e. as the expected T increases). Unless the expected time of demonopolization is zero ($\lambda = \infty$), the leader will invest later than a leader in a duopoly ($x_R > x_D$), and increasing the expected T (reducing λ) makes the trigger to converge to the monopoly trigger (x_M). The convergence is more pronounced for lower (higher) values of T (λ). If the objective of a regulator is to reduce the trigger to the leader solution in a duopoly, the effective demonopolization is the only way to achieve it.

The effect of uncertainty on the triggers for the different finite monopoly settings is shown in Figure 2. In every case a higher market uncertainty deters investment. A certain monopoly period deters more the entry of a firm, relatively to the corresponding duopoly case, the higher the uncertainty. When the level of uncertainty is small, the protection period tends not to produce any impact on the behavior of the leader, making the triggers (x_C and x_{CA}) converge to the triggers in a duopoly (x_D and x_{DA}) as uncertainty decreases. In a duopoly setting with lower uncertainty, the entry of a follower during the protection period is very unlikely, making the setting of a certain-lived monopoly similar to a duopoly in terms of the entry timing. As uncertainty increases, the trigger for investment of the granted monopolist in a random-lived monopoly (x_R) is closer to the perpetual monopolistic solution than to the duopolistic solution. This suggests that a regulator pursuing early investment should increase the threat of demonopolization for higher levels of uncertainty.

The models proposed in this paper highlight the effects of the market structure on value and firm behavior. There are significant differences between the certain-lived and the random-lived monopolies. The threat of a random change in the market structure, is only able to induce, in the most extreme case, the same behavior of a duopolistic market, with firm entry occurring somewhere between the monopoly trigger and the duopoly trigger. Giving the first firm to enter the market a certain monopoly period (instead of a uncertain/random period), produces, as the worst case, precisely the duopoly solution. Extending the monopoly protection period, induces an early entry, approaching the Marshallian trigger as that period increases. When there is no risk of preemption, firm entry can occur later than the monopoly solution if the firm operates in a certain-lived monopoly.

Using this model it is possible to obtain the optimal protection period that induces investment for a given demand level. If, for instance, a regulator or a licensor pursues investment for the current demand ($x(0)$), the optional monopoly period can be found solving $x_C = x(0)$ in order to T. Obviously that is only possible for a demand level that produces a non-negative NPV ($x(0) > x_{NPV}$), and is below the trigger for the leader in a duopoly ($x(0) < x_D$). The model also allows to compute the difference in value produced by a change in the protection period, if the licensing terms are renegotiated or the regulator decides to change the protection policy.

6 Conclusion

This paper extends the literature on investment decisions under uncertainty in different market structure settings, studying the case of a finite-lived monopoly. Under this setting a firm, idle or active in the market, benefits from being the monopolist for a finite period of protection, that ends at a random or certain date, after which the market becomes a duopoly. This demonopolization is an exogenous event, that affects the timing and value of firms.

We study three different settings for a finite-lived monopoly, and compare them with the standard monopoly and duopoly settings. When a firm benefits from a certain-lived monopoly and faces the risk of being preempted after the protection period ends, firm entry occurs sooner than in the case of a duopoly, ranging from the duopoly solution to the Marshallian trigger as the protection period increases. On the contrary, a certain-lived monopolist protected from preemption - the designated leader - will delay investment even more than in the case of a perpetual monopoly. If the duration of the monopoly protection period is random, investment occurs between the duopoly and monopoly triggers. This wide variation of the investment timing highlights the significant impact of operating in different finite-lived monopoly. We also show that uncertainty deters investment in all market structure settings.

The models presented in this paper can be extended in several ways, allowing for asymmetric firms, studying other oligopolistic markets beyond duopoly, or modelling the profit flow using different demand and production functions.

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Appendix

Proof of Proposition 1. The proof is standard. See Dixit and Pindyck (1994). □

Proof of Proposition 2. The solution to this o.d.e. corresponds to the sum of the homogeneous solution to its particular solution, for each region:[10]

$$L_R(x) = \begin{cases} b_1 x^{\eta_1} + b_2 x^{\eta_2} + L_D(x) & \text{for } x < x_F \\ \\ b_3 x^{\eta_1} + b_4 x^{\eta_2} + \dfrac{x}{r-\alpha+\lambda} + \dfrac{\lambda}{r-\alpha+\lambda}\dfrac{xD(2)}{r-\alpha} & \text{for } x \geqslant x_F \end{cases} \tag{47}$$

where b_1, b_2, b_3, and b_4 are arbitrary constants that remain to be determined, and

$$\eta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1 \tag{48}$$

$$\eta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0 \tag{49}$$

[10] Note that the value function $L_D(x)$ has two regions depending on x and x_F.

Noting that:

$$\lim_{x\to 0} L_R(x) = 0 \tag{50}$$

$$\lim_{x\to +\infty} L_R(x) = \frac{x}{r-\alpha+\lambda} + \frac{\lambda}{r-\alpha+\lambda}\frac{xD(2)}{r-\alpha} \tag{51}$$

the constants b_2 and b_3 must be set equal to zero. Condition (50) ensures that the active project is worthless if the cash flow is zero, and condition (51) reflects the expected value of L_R in a region where it is optimal for the follower to enter if the market is demonopolized (which happens with intensity λ).

For the remaining arbitrary constants two additional conditions are necessary. The two regions must met at $x = x_F$, and so $L_R(x)$ must be continuous and differentiable along x. Accordingly:

$$b_1 x_F^{*\eta_1} + L_D(x_F) = b_4 x_F^{*\eta_2} + \frac{x_F}{r-\alpha+\lambda} + \frac{\lambda}{r-\alpha+\lambda}\frac{x_F D(2)}{r-\alpha} \tag{52}$$

$$\eta_1 b_1 x_F^{\eta_1 - 1} + L'_D(x_F) = \eta_2 b_4 x_F^{\eta_2 - 1} + \frac{1}{r-\alpha+\lambda} + \frac{\lambda}{r-\alpha+\lambda}\frac{D(2)}{r-\alpha} \tag{53}$$

where $L'_D(x_F) = \dfrac{\partial L_D(x)}{\partial x}|_{x=x_F}$. Solving in order to b_1 and b_4, we get:

$$b_1 = \left(\frac{\lambda}{r-\alpha+\lambda}\frac{\eta_2-1}{\eta_1-\eta_2} + \frac{\beta_1-\eta_2}{\eta_1-\eta_2}\right)\frac{\beta_1}{\beta_1-1}Kx_F^{-\eta_1} \tag{54}$$

$$b_4 = \left(\frac{\lambda}{r-\alpha+\lambda}\frac{\eta_1-1}{\eta_1-\eta_2} + \frac{\beta_1-\eta_1}{\eta_1-\eta_2}\right)\frac{\beta_1}{\beta_1-1}Kx_F^{-\eta_2} \tag{55}$$

□

Proof of Proposition 3. Let us represent Equation (8) as $L_R(x) = L_D(x) + A(x)$ where $A(x)$ equals either $b_1 x^{\eta_1}$ (if $x < x_F$) or $b_4 x^{\eta_2} + \frac{x(1-D(2))}{r-\alpha+\lambda}$ (if $x \geqslant x_F$). The proposition verifies if $A(x) \geqslant 0$ for any given intensity rate λ. Consider the lower bound value for λ, i.e. $\lambda = 0$. In this case $b_1 = \frac{\beta_1}{\beta_1-1}\frac{1-D(2)}{D(2)}K\left(\frac{1}{x_F}\right)^{\eta_1} > 0$ and $b_4 = 0$. Since $x > 0$ we conclude that $A(x)$ is always positive, and so $L_R(x) > L_D(x)$. Consider now that $\lambda = \infty$. In this case $b_1 = b_4 = 0$ and $\lim_{\lambda\to\infty}\frac{x(1-D(2))}{r-\alpha+\lambda} = 0$, turning $A(x) = 0$ and $L_R(x) = L_D(x)$. To finally prove the proposition we need to show that b_1 and b_4 are positive for any intermediate values of λ. From Equation (11) we see that the part outside the parentheses is positive ($\beta_1 > 1$ and $D(2) < 1$) and independent from λ, so the signs of b_1 and b_4 depend on the sign of the terms inside the parentheses. For b_1 the terms inside the parentheses can be expressed as $\frac{1}{\eta_1-\eta_2}\left(-\frac{\eta_2(r-\alpha)}{r-\alpha+\lambda} - \frac{\lambda}{r-\alpha+\lambda} + \beta_1\right)$. Noting that $\eta_2 < 0$ and that $r > \alpha$ we only need to prove that $\beta_1 > 1 > \frac{\lambda}{r-\alpha+\lambda}$. Taking the limits we see that $\frac{\lambda}{r-\alpha+\lambda} \in (0,1)$, so $b_1 > 0$, for any $\lambda \in (0,+\infty)$. As far as b_4 is concerned, we only need to show that $\beta_1 > \eta_1$ for any $\lambda > 0$. This is an obvious result after considering Equations (6) and (9). □

Proof of Proposition 4. The proof is straightforward after the proof of Proposition 3. □

Proof of Proposition 5. The steps for the proof are straightforward, based the proof of Proposition 3. Notice that $\eta_1 = \beta_1$ when $\lambda = 0$ and that the only relevant region for the value function $L_R(x)$ is the one represented in the first branch of Equation (3). □

Proof of Proposition 6. The general solution to the o.d.e. (18) is $O_R(x) = a_1 x^{\eta_1} + a_2 x^{\eta_2}$ for $x < x_D$ and $O_R(x) = a_3 x^{\eta_1} + a_4 x^{\eta_2}$ for $x_D \leqslant x < x_R$. Given that $\lim_{x\to 0} O_R(x) = 0$, a_2 must be set to 0. The solution corresponds to the sum of the homogeneous solution to the particular solution for each region. □

Proof of Proposition 7. From Proposition 4 we know that, when $\lambda = \infty$, $L_R(x) = L_D(x)$ and so, necessarily, $x_D = x_R$. For this reason, the second branch of the Equation (20) collapses when $\lambda = \infty$ ($x_R \to x_D$ as $\lambda \to \infty$). To determine a_1, consider the value matching condition: $a_1 x_D^{\eta_1} + F_D(x_D) = L_R(x_D) - K$. By definition, the leader net value function meets the follower value function at $x = x_D$, which means that there is no additional value for being the leader at that moment. Consequently, a_1 must be set equal to zero. □

Proof of Proposition 8. Based on Proposition 5, the third branch of Equation (20) becomes $\frac{x}{r-\alpha} - K$. Additionally, when $\lambda = 0$ the entrance of a follower will never be possible, so this role is worthless ($F_D(x) = 0$). The idle firm, facing no threat of demonopolization, will defer the investment decision until the threshold (x_M) is achieved. At that moment the value of the idle firm matches the NPV, so the interval for the second branch disappears. After computing for a_1 and x_M based on the value-matching and smooth-pasting conditions we easily get Equation (16). □

Proof of Proposition 9. In order to derive the value of the forward start follower we need to value the discounted risk-neutral expected follower value:

$$F_C(x) = e^{-rT} E\left[F_D(x(T))\right] \tag{56}$$

where $F_D(x(T))$ is the value of the follower's option to invest at time T. Note that from Equation (4) $F_D(x(T))$ has two regions, and so:

$$F_C(x) = e^{-rT} E\left[\left(\frac{x(T)(1-D(2))}{r-\alpha} - K\right)\mathbf{1}_{x(T)\geqslant x_F}\right] + e^{-rT} E\left[g x(T)^{\beta_1}\mathbf{1}_{x(T)<x_F}\right] \tag{57}$$

where $\mathbf{1}_{\text{condition}}$ equals 1 if the condition is met, and 0 otherwise, and $g = \frac{K}{\beta_1 - 1}\left(\frac{1}{x_F}\right)^{\beta_1}$.

Following Shackleton and Wojakowski (2007, p. 3849-50), the first component in $F_C(x)$ is the difference between an asset-or-nothing call option on $\frac{x(1-D(2))}{r-\alpha}$, and a cash-or-nothing call option on K, with exercise prices x_F, and maturity T:

$$e^{-rT}E\left[F_D(x(T))\right]\mathbf{1}_{x(T)\geqslant x_F} = \frac{x(1-D(2))}{r-\alpha}e^{-(r-\alpha)T}N\left(d_1(x)\right) - Ke^{-rT}N\left(d_2(x)\right) \tag{58}$$

where

$$d_1(x) = \frac{\ln\left(\frac{x}{x_F}\right) + \left(\alpha + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{59}$$

$$d_2(x) = d_1(x) - \sigma\sqrt{T} \tag{60}$$

$$\tag{61}$$

From the Appendix A in Shackleton and Wojakowski (2007), the second component in $F_C(x)$ equals:

$$e^{-rT}E\left[F_D(x(T))\right]\mathbf{1}_{x(T)<x_F} = gx^{\beta_1}N\left(-d_3(x)\right) = \frac{K}{\beta_1 - 1}\left(\frac{x}{x_F}\right)^{\beta_1}N\left(-d_3(x)\right) \tag{62}$$

where

$$d_3(x) = \frac{\ln\left(\frac{x}{x_F}\right) + \left(\alpha + \left(\beta_1 - \frac{1}{2}\right)\sigma^2\right)T}{\sigma\sqrt{T}} \tag{63}$$

$$d_3(x) = d_1(x) + (\beta_1 - 1)\,\sigma\sqrt{T} \tag{64}$$

□

Proof of Proposition 10. Since $\lim_{T\to\infty} N\left(-d_3(x)\right) = 0$, $\lim_{T\to\infty} e^{-(r-\alpha)T} = 0$, and $\lim_{T\to\infty} e^{-rT} = 0$, it follows that $\lim_{T\to\infty} F_C(x) = 0$. □

Proof of Proposition 11. When $x < x_F \lim_{T\to 0}, N\left(-d_3(x)\right) = 1$, $\lim_{T\to 0} N\left(d_1(x)\right) = 0$, and $\lim_{T\to 0} N\left(d_2(x)\right) = 0$, and it follows that $\lim_{T\to 0} F_C(x) = F_D(x)$ (upper branch of Equation (4)). When $x \geqslant x_F, \lim_{T\to 0} N\left(-d_3(x)\right) = 0$, $\lim_{T\to 0} N\left(d_1(x)\right) = 1$, and $\lim_{T\to 0} N\left(d_2(x)\right) = 1$, and it follows that $\lim_{T\to 0} F_C(x) = F_D(x)$ (lower branch of Equation (4)). □

Proof of Proposition 12. The value of an active leader that benefits from a certain monopoly period can be decomposed as follows:

$$L_C(x) = \frac{x}{r-\alpha} - e^{-rT}E\left[\left(\frac{x(T)(1-D(2))}{r-\alpha}\right)\mathbf{1}_{x(T)\geqslant x_F}\right] - e^{-rT}E\left[hx(T)^{\beta_1}\mathbf{1}_{x(T)<x_F}\right] \tag{65}$$

where $h = \frac{\beta_1}{\beta_1 - 1}K\left(\frac{1}{x_F}\right)^{\beta_1}$.

The value of a leader in this setting, equals the value of a monopolist (first component) deducted from the cost of having to share the market. That happens at time T if the follower trigger is reached ($x(T) \geqslant x_F$), or later than T, otherwise ($x(T) < x_F$). The latter is similar to the case of a duopoly, conditional on $x(T) < x_F$, and therefore h is obtained from Equation (3).

From the Appendix A in Shackleton and Wojakowski (2007), the last component in $L_C(x)$ equals:

$$e^{-rT}E\left[hx(T)^{\beta_1}\mathbf{1}_{x(T)<x_F}\right] = hx^{\beta_1}N(-d_3(x)) = \frac{\beta_1}{\beta_1-1}K\left(\frac{x}{x_F}\right)^{\beta_1}N(-d_3(x)) \tag{66}$$

The second component in $L_C(x)$ is an asset-or-nothing call option on $\frac{x(1-D(2))}{r-\alpha}$, with an exercise price x_F, and maturity T:

$$e^{-rT}E\left[\left(\frac{x(T)(1-D(2))}{r-\alpha}\right)\mathbf{1}_{x(T)\geqslant x_F}\right] = \frac{x(1-D(2))}{r-\alpha}e^{-(r-\alpha)T}N(d_1(x)) \tag{67}$$

□

Proof of Proposition 13. Since $\lim_{T\to\infty} N(-d_3(x)) = 0$, and $\lim_{T\to\infty} e^{-(r-\alpha)T} = 0$, it follows that $\lim_{T\to\infty} L_C(x) = x/(r-\alpha)$. x_C the solution to $f(x_C) = L_C(x_C) - K - F_C(x_C) = 0$. Using Proposition 10, $\lim_{T\to\infty} f(x_C) = x_C/(r-\alpha) - K$, and $\lim_{T\to\infty} x_C = (r-\alpha)K$. □

Proof of Proposition 14. When $x < x_F, \lim_{T\to 0} N(-d_3(x)) = 1$, and $\lim_{T\to 0} N(d_1(x)) = 0$, and it follows that $\lim_{T\to 0} L_C(x) = L_D(x)$ (upper branch of Equation (3)). When $x \geqslant x_F, \lim_{T\to 0} N(-d_3(x)) = 0$, and $\lim_{T\to 0} N(d_1(x)) = 1$, and it follows that $\lim_{T\to 0} L_C(x) = L_D(x)$ (lower branch of Equation (3)). Given that $\lim_{T\to 0} L_C(x) = L_D(x)$, and $\lim_{T\to 0} F_C(x) = F_D(x)$ (Proposition 11), the trigger x_C must be the same as the duopoly trigger, x_D, for $T = 0$. □

Proof of Proposition 15. The value function for the pre-assigned leader, $O_{DA}(x)$, must satisfy the following non-homogeneous o.d.e:

$$\frac{1}{2}\sigma^2x^2\frac{\partial^2 O_{DA}(x)}{\partial x^2} + \alpha x\frac{\partial O_{DA}(x)}{\partial x} - rO_{DA}(x) = 0 \tag{68}$$

The general solution to this o.d.e. is $O_{DA}(x) = d_1 x^{\beta_1} + d_2 x^{\beta_2}$. Given that $\lim_{x\to 0} O_{DA}(x) = 0$, d_2 must be set to 0. The following value matching and smooth-pasting boundary conditions allows us to obtain x_{DA} and d_1:

$$d_1 x_{DA}^{\beta_1} = L_D(x_{DA}) - K \tag{69}$$

$$\beta_1 d_1 x_{DA}^{\beta_1 - 1} = \frac{\partial L_D(x)}{\partial x}|_{x=x_{DA}} \tag{70}$$

Smooth pasting is not possible for $x > x_F$ and, therefore, using the upper branch of Equation (3), these conditions become:

$$d_1 x_{DA}^{\beta_1} = \frac{x_{DA}}{r-\alpha} - \frac{\beta_1}{\beta_1 - 1} K \left(\frac{x_{DA}}{x_F}\right)^{\beta_1} - K \tag{71}$$

$$\beta_1 d_1 x_{DA}^{\beta_1 - 1} = \frac{1}{r-\alpha} \tag{72}$$

Solving these equations we obtain:

$$d_1 = \frac{K}{\beta_1 - 1} \left(\frac{1}{x_{DA}}\right)^{\beta_1} - \frac{\beta_1}{\beta_1 - 1} K \left(\frac{1}{x_F}\right)^{\beta_1} \tag{73}$$

$$x_{DA} = \frac{\beta_1}{\beta_1 - 1}(r-\alpha)K = x_M \tag{74}$$

□

Proof of Proposition 16. The value function for the pre-assigned leader, $O_{CA}(x)$, must satisfy the following non-homogeneous o.d.e.:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 O_{CA}(x)}{\partial x^2} + \alpha x \frac{\partial O_{CA}(x)}{\partial x} - r O_{CA}(x) = 0 \tag{75}$$

The general solution to this o.d.e. is $O_{CA}(x) = c_1 x^{\beta_1} + c_2 x^{\beta_2}$. Given that $\lim_{x\to 0} O_{CA}(x) = 0$, c_2 must be set to 0. The following value matching and smooth-pasting boundary conditions allows us to obtain x_{CA} and c_1:

$$c_1 x_{CA}^{\beta_1} = L_C(x_{CA}) - K \tag{76}$$

$$\beta_1 c_1 x_{CA}^{\beta_1 - 1} = \frac{\partial L_C(x)}{\partial x}|_{x=x_{CA}} \tag{77}$$

Substituting $L_C(x_{CA})$ with Equation (36), the second equation becomes:

$$\beta_1 c_1 x_{CA}^{\beta_1} = \frac{x}{r-\alpha} - x\frac{\partial}{\partial x}\left[\frac{x\,(1-D(2))}{r-\alpha}e^{-(r-\alpha)T}N\left(d_1(x)\right)\right] - x\frac{\partial}{\partial x}\left[\frac{\beta_1}{\beta_1-1}K\left(\frac{x}{x_F}\right)^{\beta_1}N\left(-d_3(x)\right)\right] \tag{78}$$

The solutions to the derivatives can be found in Shackleton and Wojakowski (2007, section 4). Substituting $c_1 x_{CA}^{\beta_1}$ with Equation (76), after simplification the following equation is obtained:

$$\frac{x_{CA}}{r-\alpha}\left(1-(1-D(2))\,e^{-(r-\alpha)T}N\left(d_1(x_{CA})\right)\right) - \frac{\beta_1}{\beta_1-1}K = 0 \tag{79}$$

x_{CA} is the numerical solution to this equation. c_1 is obtained using Equation (76), and can be expressed as $c_1 = \left(L_C(x_{CA}) - K\right)x_{CA}^{-\beta_1}$. □

Proof of Proposition 17. Since $\lim_{T\to\infty} e^{-(r-\alpha)T} = 0$, in the limit Equation (43) reduces to $\frac{x_{CA}}{r-\alpha} - \frac{\beta_1}{\beta_1-1}K = 0$. Solving this equation we obtain $x_{CA} = x_M$. □

Proof of Proposition 18. Since $\lim_{T\to 0} N\left(d_1(x)\right) = 0$, in the limit Equation (43) reduces to $\frac{x_{CA}}{r-\alpha} - \frac{\beta_1}{\beta_1-1}K = 0$. Solving this equation we obtain $x_{CA} = x_M = x_{DA}$. □

Proof of Proposition 19. Denoting $f(x) = \frac{x}{r-\alpha}\left(1-(1-D(2))\,e^{-(r-\alpha)T}N\left(d_1(x)\right)\right) - \frac{\beta_1}{\beta_1-1}K$, x_{CA} is the solution to $f(x) = 0$ (Equation (43)). It is possible to show that $f(x)$ is increasing in x and that $f(x_{DA}) < 0$, and therefore $x_{CA} > x_{DA}$. In fact, since $\frac{\partial f(x)}{\partial x} = \frac{1}{r-\alpha}\left(1-(1-D(2))\,e^{-(r-\alpha)T}N\left(d_1(x)\right)\right)$ (see proof of Proposition 16), $1-D(2) < 1$, $e^{-(r-\alpha)T} < 1$, and $N\left(d_1(x)\right) < 1$, $\frac{\partial f(x)}{\partial x} > 0$. From Proposition 15, $x_{DA} = \frac{\beta_1}{\beta_1-1}(r-\alpha)K$, it follows that $\frac{x_{DA}}{r-\alpha} - \frac{\beta_1}{\beta_1-1}K = 0$, and since $1-D(2) > 0$, $e^{-(r-\alpha)T} > 0$, and $N\left(d_1(x_{DA})\right) > 0$, $f(x_{DA}) < 0$. □